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Children crying at birthday parties. Why? Fairness and incentives for cake division problems

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Children crying at birthday parties. Why? Fairness and incentives for cake division problems

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Abstract

We consider the problem of dividing a non-homogeneous one-dimensional continuum whose endpoints are topologically identified. Examples are the division of a birthday cake, the partition of a circular market, the assignment of sentry duty or medical call. We study the existence of rules satisfying various requirements of fairness (no-envy, egalitarian-equivalence; and several requirements having to do with changes in the data of the problem), and that induce agents to reveal their preferences honestly (strategy-proofness).

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1 Introduction

A circular birthday cake has to be divided among a group of children. Slices are cut along radii, and each child is to receive a slice. Frosting and decorations are distributed unevenly on the cake. Children have different preferences over slices, except that of two slices related by inclusion, each child prefers the larger one. How should the division be performed?

A road around a lake in a mountain resort is to be divided into territories or markets among ice-cream or souvenir vendors. The road is lined with towns and beaches. The potential customers of these vendors are distributed unevenly along the road. Vendors have different preferences over markets, due to what they sell, the equipment they operate, the time it takes them to drive to where they would set up business, and so on, but of two markets related by inclusion, each vendor prefers the larger one. What partition should be selected?

More generally, the issue is to partition into intervals an infinitely divisible, non-homogeneous, and atomless one-dimensional continuum whose endpoints are topologically identified. The recipients are equipped with preferences over intervals that are continuous and monotonic with respect to interval-inclusion.

Our analysis covers the case of continuous preferences such that, of two intervals related by inclusion, the smaller one is preferred to the larger one. The issue might be that of dividing among children a quiche with uneven distributions of mushrooms and spinach, or assigning sections of a road around a lake to policemen to patrol, or to maintenance crews to clean. Then, an enlargement of an agent's assignment would cause his welfare to decrease.

The problem has a time-division interpretation too. Think of a facility that has to be staffed around the clock (emergency room in a hospital; fire station; military installation; plant in which an industrial process that runs continuously must be monitored). The search here is for a rotation of personnel that satisfies this staffing requirement. In these examples, of two intervals ordered by inclusion, each agent prefers the smaller one to the larger one, but one could also imagine situations where the opposite holds.

We enquire about the existence of partitions satisfying various normative requirements of efficiency and fairness, and about the existence of division rules satisfying in addition one of several normative or strategic relational requirements. Efficiency of a rule is the usual requirement that for each partition it selects, there should not be another partition that is unanimously preferred to it. For fairness, we consider what have arguably been the two main requirements in the literature, no-envy and egalitarian-equivalence. The relational requirements express ideas of solidarity on the one hand, and robustness under manipulation on the other.

We also examine a version of the model in which each agent is endowed with an interval, the profile of endowments defining a partition. This partition may not be deemed socially desirable—perhaps it is not efficient—and the issue then is to find a better one, taking initial holdings into account.

We offer positive and negative results. Difficulties occur with no-envy. In particular, preferences may be such that no envy-free and efficient partition exists. In such a situation, a well-meaning parent insisting on selecting a partition of a birthday cake to which no other would be preferred by all children, necessarily produces one at which at least one child prefers the slice assigned to some other child to his or her own. It is no wonder that children are sometimes seen to cry at birthday parties.

No such difficulty occurs with egalitarian-equivalence, and in fact selections from the intersection of that solution with the Pareto solution can easily be defined that satisfy various solidarity properties.

Incentive properties are quite demanding as soon as efficiency is required, and whether or not fairness requirements are imposed, a conclusion that reinforces the lesson that one can draw from the literature on mechanism design of the last thirty years.

2 Related literature

An extensive literature exists on the "cake division problem", to which mathematicians have been the main contributors. There, a cake is modelled as a compact subset of some Euclidean space and agents are equipped with additive preferences over measurable subsets. Initiated by Steinhaus (1948), it focuses almost exclusively on achieving some notion of equity, and no constraints are usually imposed on the shapes of partitions (an exception is Hill, 1983). We refer to Brams and Taylor (1996), Robertson and Webb (1998), and Barbanel (2005) for detailed accounts of the large recent literature. Efficiency is considered in this new literature, an important characteristic of which is that it places much emphasis on algorithmic procedures to identify desirable partitions, whereas economists are often satisfied with results stating the existence of such partitions.

The special case of a one-dimensional non-homogeneous and atomless continuum, when each agent is to receive an interval, has been examined by several authors (Stromquist, 1980; Weller, 1985; Berliant, Dunz and Thomson, 1992; and Thomson, 2003). We refer to it as the **interval division problem**. The circular variant, which is the focus of our study, had so far not been the object of systematic analysis. The one-dimensional linear model and the one-dimensional circular model seem to be closely related, but an important conclusion to be drawn from the literature just cited and the current paper, is that whether on not the continuum closes on itself makes an important difference.

We should also note that, although the restriction that preferences be additive, which is common, is an interesting one—and in fact, we prove a number of results on the additive domain—our preferences are allowed to be more general, and in particular they may exhibit complementarities between parts of the dividend. The recent paper by Barbanel and Brams (2005) concerns the additive case. When preferences can be represented by measures, it is tempting to use as an index of an agent's welfare at a partition the measure that he assigns to his component of the partition, and to require that, using these indices, welfares should be equal across agents. Barbanel and Brams propose for two agents a division procedure designed to yield partitions along diameters, and for three agents, one that yields partitions of the kind discussed in the current paper. They discuss whether the procedures satisfy no-envy, the equal-measures criterion, and efficiency (for the diametric procedure, efficiency is defined relative to diametric divisions). We discuss other aspects of their contribution below.

Finally, we comment on the recent work on "queueing", "sequencing", and "scheduling" (Suijs, 1996; Maniquet, 2003; Moulin, 2004; Chun, 2004). In this literature, preferences belong to parametric classes and monetary compensations are possible. Then, some rules exist that satisfy appealing normative and some strategic properties. The possibility of monetary compensation is also considered by Brams, Jones, and Klamler (2003).

3 The model

Let $N \equiv \{1, \ldots, n\}$ be a set of agents among whom a closed curve has to be divided into intervals. Without loss of generality, we think of the curve as a circle and we refer to the intervals as arcs. Each agent $i \in N$ is equipped with a preference relation over arcs denoted R_i. This relation is continuous; it attaches no value to individual points, so that we can assume that all arcs are closed; and it is monotone in the sense that, given two (closed) arcs related by inclusion, the larger one is preferred to the smaller one. Let $\mathcal{R}$ be the class of these preference relations and $\mathcal{R}^N$ the $|N|$-fold Cartesian product of $\mathcal{R}$ with itself. A **problem** is a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$. The circle has to be divided into arcs, one for each agent. These arcs should not overlap and together they should cover the entire circle. We call such divisions **partitions**, slightly abusing mathematical language since adjacent arcs have at least one endpoint in common.[1] This modelling choice is made possible by our assumption on preferences that arcs of zero length have no value. Also, we allow empty arcs. Let $\boldsymbol{A}$ be the set of arcs and $\boldsymbol{X}$ the set of partitions.

We search for well-behaved methods of associating with each problem a non-empty set of partitions.

Definition A **solution** is a correspondence $\varphi\colon \mathcal{R}^N \rightarrow X$ that associates with each problem $R \in \mathcal{R}^N$ a non-empty subset of X, denoted $\varphi(R)$. Given $x \in \varphi(R)$ and $i \in N$, x_i is the arc received by agent i. A **rule** is a single-valued solution.

More generally, one could allow agents to receive unions of intervals, and for some applications, this certainly would be required. In other applications however, it is completely natural to insist that each agent should receive a single interval. Returning to the examples mentioned in the introduction, when the problem is to assign sections of a road around a lake to policemen to patrol, it would indeed be strange to assign to one of them several disconnected sections. When the issue is to partition time around the clock between engineers to handle emergencies that may occur in the operation of some industrial process that runs continuously, assigning to an engineer several distinct time intervals over a 24-hour period will often not make sense, although excluding the possibility would probably not be as compelling.

Allowing consumptions to be unions of intervals would require that preference relations be defined over such unions. Thus, they would become much more complicated objects. A two-dimensional representation of the problem of the kind that we develop below would not be an option anymore. Even if each agent were limited to receiving a union of at most two intervals, the next

[1] They have two in the two-agent case if both agents receive arcs of positive length.

most complicated case, a four-dimensional space would be needed. Thus, we will not discuss these more general situations. However, it is intuitive that the negative results that we present would extend because the scope of the requirements on which they are based would be wider.

If a rule always selects a partition in the image of a particular solution, it is a **selection** from the solution.

4 Arcs, partitions, and preferences: a two-dimensional geometric representation

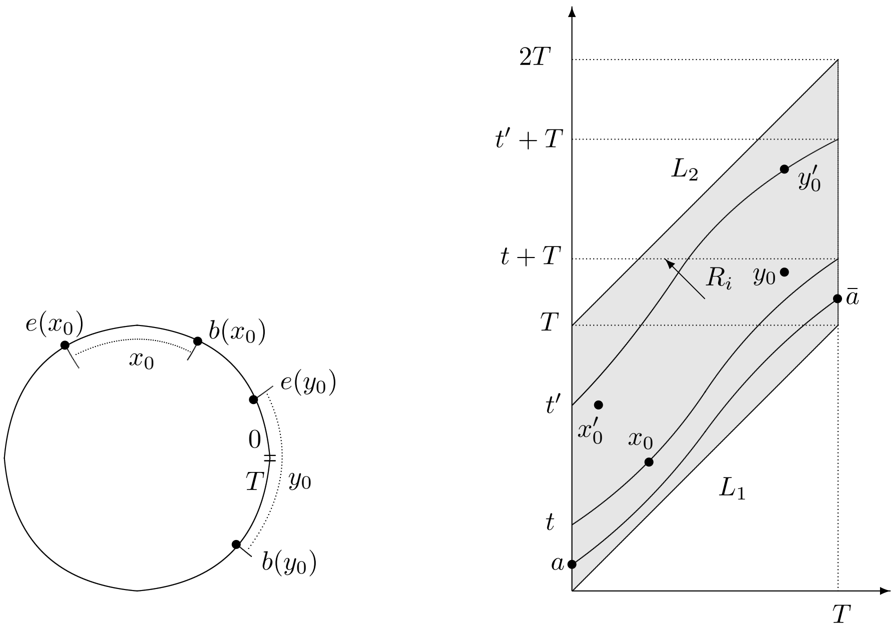
A significant part of this paper is devoted to developing a geometric representation of the feasible set and of preferences in two-dimensional Euclidean space. We believe that the reader's patience will be rewarded however, as this representation will allow intuitive, elementary, and almost entirely self-contained resolutions of many of the issues that we will address.

Arcs. We index the points of the circle by a parameter t running counterclockwise from some arbitrarily chosen origin 0 to some maximal value called T (Figure 1a). We identify an arc by its endpoints. An arc can begin anywhere in $[0, T]$ and be of any length in $[0, T]$. Given $x_0 \in A$, let $b(x_0) \in [0, T]$ be the point where x_0 begins and $e(x_0) \in [0, 2T]$ the point where it ends. By definition, $b(x_0) \leq e(x_0)$. Thus, x_0 is represented by the vector $(b(x_0), e(x_0))$ in the non-negative quadrant of a two-dimensional Euclidean space. This vector is such that $e(x_0) - b(x_0) \leq T$. The set of arcs is the convex hull of the four points $(0, 0)$, (T, T), $(T, 2T)$, and $(0, T)$, the shaded region of Figure 1b.[2]

Notation: In $\mathbb{R}^2$, the segment connecting points a and b is denoted $\mathrm{seg}[a, b]$. To exclude endpoint a, say, from the segment connecting a to b, we write $\mathrm{seg}\,]a, b]$.

Preferences. Preference relations are defined over A. An example of a preference relation for $i \in N$ is illustrated in Figure 1b. All preference relations in $\mathcal{R}$ share the following features:

[2]Alternative representations are possible. An arc could be indexed by (i) the point where it begins and its length, (ii) the point where it ends and its length, (iii) its midpoint and its length.



(a) (b)

Figure 1: Partitioning a circle into arcs. (a) Arcs are measured counterclockwise from the origin marked 0. (b) Representation in $\mathbb{R}^2_+$ of a preference relation over arcs.

1. All the points of $\boldsymbol{L_1} \equiv \text{seg}[(0,0),(T,T)]$ represent arcs of length 0, so they are indifferent to each other.

2. All the points of $\boldsymbol{L_2} \equiv \text{seg}[(0,T),(T,2T)]$ represent the entire circle, so they too are indifferent to each other.

3. For each $t \in [0,T]$, the point $(0,t)$ represents the same arc as the point $(T, t+T)$. Thus, if an indifference curve reaches the vertical axis at a point of ordinate t, it reaches the vertical line of abscissa T at a point of ordinate $t+T$. These points are denoted by the same letter, with and without an upper bar. An example is $\{a, \bar{a}\}$ in Figure 1b.

4. Given two arcs related by inclusion, x_0 and x'_0, the larger one is represented by a point to the northwest of the smaller one. Thus, the direction of increasing welfare is northwesternly, and indifference curves are strictly upward sloping (upward sloping with no horizontal or vertical segments).

Continuity of preferences and items 1 and 2 in the list above imply that indifference curves passing through points that are close to L_1 are close to being segments of slope 1. Similarly, indifference curves passing through points that are close to L_2 are close to being segments of slope 1.

Here are examples of maps. In addition to helping us gain insight into the nature of the problem, they will be useful in certain proofs.

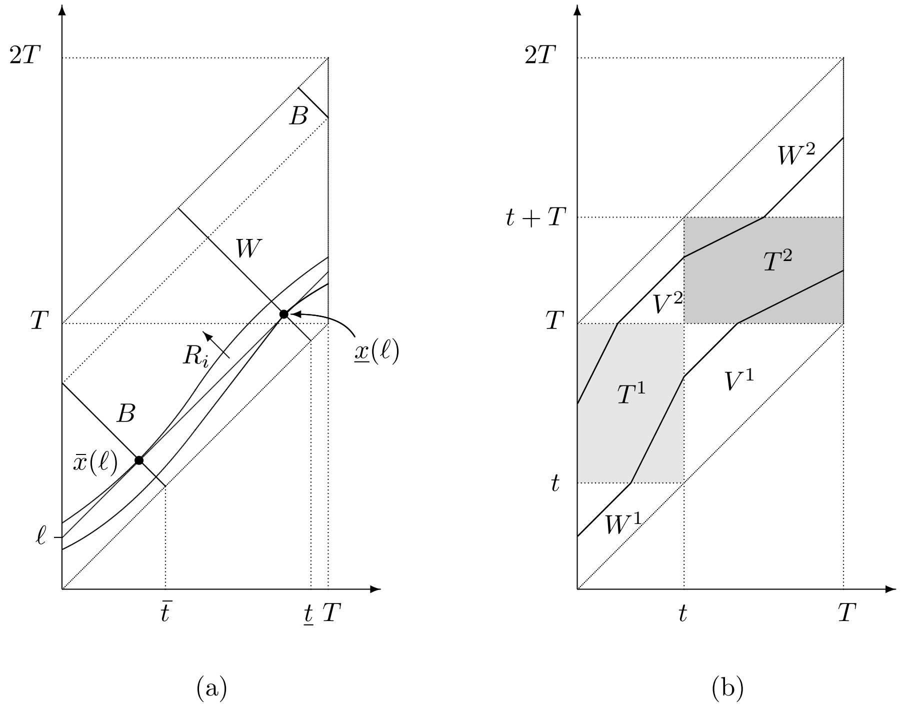


Figure 2: Special maps. (a) A notion of single-peaked preferences. (b) Additive preferences relative to a two-component partition of the circle.

• **When only the length of an arc matters.** Consider someone who only cares about the length of the arc he receives. His indifference map consists of segments of slope 1 from the vertical axis to the vertical line of abscissa T. Then, the good can be thought of as being homogeneous.

• **Single-peaked preferences.** A natural notion of single-peaked preferences can be defined as follows (Figure 2a): for each $\ell \leq T$, there is a unique most preferred arc of length ℓ, $\bar{x}(\ell)$, and arcs obtained by rotating $\bar{x}(\ell)$ counterclockwise are worse and worse until some worst arc of that length, $\underline{x}(\ell)$; similarly, arcs obtained by rotating $\bar{x}(\ell)$ clockwise are worse and worse until that worst arc. An example is when indifference curves have the concave-convex shape suggested in Figure 2a. But single-peakedness is more general. What matters is simply that along lines of slope 1 in arc space, welfare varies as described above.

An interesting further restriction is when there is $\bar{t} \in [0, T]$ such that for each $\ell \leq T$, the center of $\bar{x}(\ell)$ is $\bar{t}$. The point $\bar{t}$ can be interpreted as an "ideal" location for the agent since for each interval length, the optimal interval of that length is centered at $\bar{t}$. Similarly, there may be a point $\underline{t} \in [0, T]$ that represents a worst location. This assumption is illustrated in Figure 2a, as the locus of the maximizer of the relation on the segment representing all the arcs of length ℓ, as ℓ varies from 0 to T, is a segment of slope -1 (namely the segment B, which is in two parts). Similarly, the locus of the minimizer of the relation on these segments is a segment of slope -1 (the segment W).

Additionally, we could require that for each $\ell \leq T$, two arcs obtained from $\bar{x}(\ell)$ by the same rotation, counterclockwise and clockwise, be indifferent. (This assumption corresponds to the symmetric single-peaked case in social choice theory.)

• **Additive preferences.** To motivate the next restriction, "additivity", we return to the interpretation of the circle as a market. Suppose that arcs are valuable to agents because of customers distributed along them and that profit per customer is constant. Then, preferences have additive numerical representations (Figure 2b). Here is an illustration when the circle can be partitioned into two homogeneous parts, with a uniform density of customers in each. We exploit this construction in the proof of Theorem 8. Let $t \in T$ and $k > 0$. Let $A^1 \equiv [0, t]$ and $A^2 \equiv [t, T]$, each subarc of A^2 being k times as valuable as each subarc of A^1 of the same length. We show next that arc space can then be partitioned into regions in each of which indifference curves are parallel segments.[3]

Here is how to identify these regions and the slopes of these segments. In Figure 2b, the vertical line of abscissa t and the horizontal lines of ordinates t and $t{+}T$ partition arc space into six regions, four triangles and two rectangles. These regions are labelled in the figure. In each of the triangles, indifference curves are segments of slope 1. Indeed, any two points on a segment of slope 1 in either V^1 or W^1 represent two subintervals of A^1 of the same length (for W^1) or two subintervals of A^2 of the same length (for V^1). Also, any two points on a segment of slope 1 in either V^2 or W^2 represent two superintervals of A^2 of the same length (for W^2) or two superintervals of A^1 of the same length (for V^2). Each point in rectangle T^1 represents an arc whose starting point is in A^1 and endpoint is in A^2. If we denote by δ the density of customers in A^1, rotating an arc counterclockwise by some amount τ such that the starting point remains in A^1 and the endpoint remains in A^2 means losing $\delta\tau$ customers in A^1 and gaining $\delta\tau k$ customers in A^2. Thus, in that rectangle, indifference curves are segments of slope $\frac{1}{k}$. By a similar argument, in rectangle T^2, indifference curves are segments of slope k.

A partition of the circle into more than two regions of uniform customer density induces a finer partition of the set of feasible arcs. An example of a three-component partition is represented in Figure 6b below. More generally, consider a finite partition with $0 = a_0, a_1, \ldots, a_{k-1}, a_k = T$ being

[3] Additive preferences arising from a two-component partition of the circle can be thought of as "single-plateaued".

the successive boundary points of its components. Agent i's preferences can then be represented by the function $u_i\colon A \to \mathbb{R}$ defined by $u_i(x) \equiv \sum_{\ell=0}^{\ell=k} \ell(x \cap [a_\ell, a_{\ell+1}])$, where $\ell(x_0)$ refers to the length of arc x_0. The segments L_1 and L_2 are still lined with triangles in which indifference curves are segments of slope 1 and the remainder of arc space is partitioned into rectangles that can be matched in pairs, as follows: for each pair $\{i, j\}$ with $0 \leq i < j < k$, indifference curves in $[a_i, a_{i+1}] \times [a_j, a_{j+1}]$ are segments whose slopes are equal to the ratio of customer densities in arcs $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$; indifference curves in $[a_j, a_{j+1}] \times [a_i + T, a_{i+1} + T]$ are segments whose slopes are the inverse ratio.

The most general additive case is when there is an integrable function defined along the circle, and arcs are ranked according to the integrals of the function along them.[4]

• **Smooth preferences.** Smooth preferences have continuous differentiable numerical representations. Geometrically, this means that indifference curves have no kinks. In particular, at the point $a \equiv (0, \ell)$ where an indifference curve reaches the vertical axis, the curve has a slope that equals its slope at the point $\bar{a} \equiv (T, \ell + T)$ where it reaches the vertical line of abscissa T, since a and $\bar{a}$ represent the same arc. The highest indifference curve of the map represented in Figure 1b looks smooth but in fact, smoothness is violated because its slope at $(0, t')$ is greater than 1 and its slope at $(T, t' + T)$ is smaller than 1. If arcs were measured from a different origin along the circle, this difference in slopes would show up as a kink for the new map.

• **Convex preferences.** Convexity of preferences can be defined in the usual way, as convexity of upper contour sets. However, whether or not a relation is convex depends on the choice of origins. Only one relation has upper contour sets that are convex for each choice of origin, namely the trivial relation for which the good is homogeneous. Of course, in some applications, "convexity for a particular choice of origin" may be economically meaningful. For instance, when the dividend is interpreted as a road around a lake, and if the origin is chosen to be the point where the road is reached from the neighboring town, where the vendors live, this limited notion of convexity

[4]Other interesting preferences can be defined. The case of "atoms" can also be accommodated by our geometric representation, but there are unavoidable complications. Indeed, it becomes necessary to specify whether the endpoints of an interval assigned to an agent are included in his assignment. Each point in the interior of arc space comes with four different interpretations, depending upon whether its endpoints are included. The points of seg$[(0, 0), (T, T)]$ and seg$[(0, T), (T, 2T)]$ have only two interpretations however.

makes sense.

Partitions. At a partition, where one agent's arc ends is where someone else's arc begins. Thus, a partition is a sequence of arcs indexed by agents, $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$, such that $e(x_{i_1}) = b(x_{i_2})$, $e(x_{i_2}) = b(x_{i_3})$, $\ldots$, $e(x_{i_n}) = b(x_{i_1}) + T$. In $\mathbb{R}^2_+$, a partition is then represented as a set of points arranged in an increasing sequence. By drawing horizontal and vertical lines through them, one defines a "staircase", of generally uneven steps, whose inside kinks belong to the 45° line. The construction is illustrated in Figure 3a for the two-agent case and in Figure 3b for the four-agent case. Except when one agent receives the entire circle, only one agent's arc is represented by a point of ordinate greater than T.[5] Indeed, only one arc in a partition can contain the origin in its interior. (In the exceptional case just mentioned, the partition is represented by any point of L_1 together with any point of L_2.) partition, by the "first arc", we mean the arc whose starting point is the closest to the origin, measuring distances counterclockwise, and by the "last arc", the arc whose starting point is the furthest from the origin. Where the last arc ends is where the first arc begins, so in the figures, the vertical line through the first arc and the horizontal line through the last arc meet on L_2.

A variant of the model is obtained by imagining that initially, each agent is endowed with an arc, this profile of endowments defining a partition. Denoting by $\omega_i \in A$ agent i's endowment and $\omega \equiv (\omega_i)_{i \in N} \in X$ the endowment profile, a **problem with endowments** is a pair $(R, \omega) \in \mathcal{R}^N \times X$. The issue here is how to redistribute these endowments. The endowment profile may represent initial ownerships. For instance, it may be a partition chosen in some previous occurrence of the situation. In the application to the division of a market into territories, it could be last year's partition. It may be felt that this partition is relevant to this year's problem, perhaps that it should be used as a benchmark, for example that it should serve to define minimal welfare levels for agents.

5 Efficiency

We now turn to the search for well-behaved solutions. We start with the solution that is at the center of modern microeconomics. It associates with each problem its set of allocations (here, partitions) having the property that

[5]When the origin is the common boundary point of two arcs, no arc does.

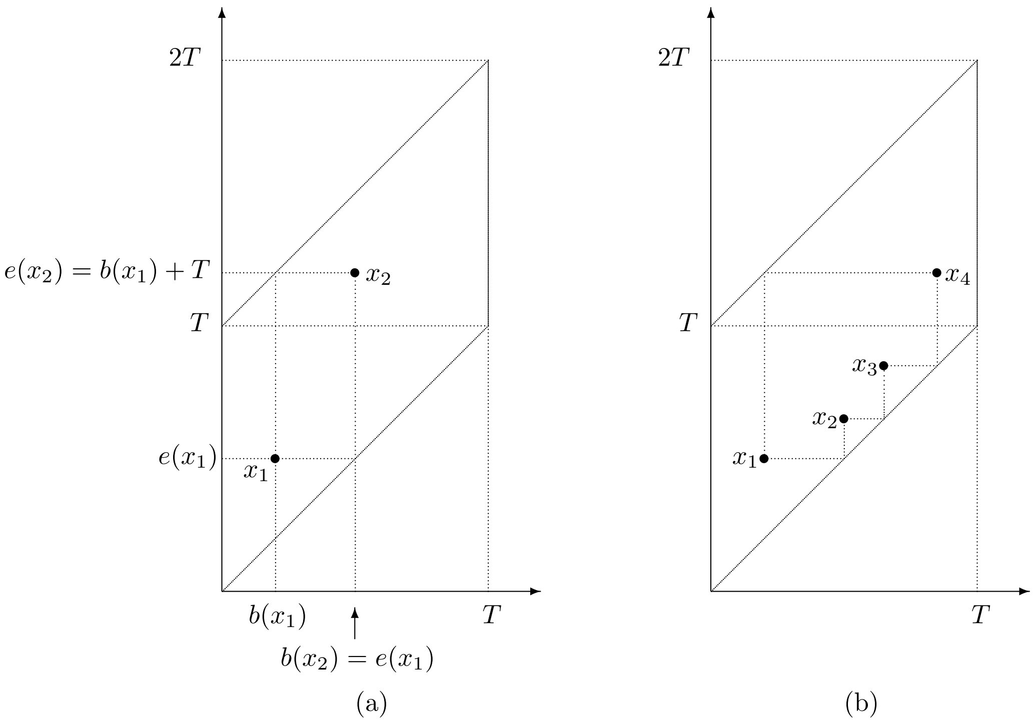


Figure 3: Partitions. (a) Two-person example. (b) Four-person example.

there is no other allocation that each agent finds at least as desirable and at least one agent prefers:[6]

Pareto solution, *P*: For each $R \in \mathcal{R}^N$, $P(R) \equiv \{x \in X$: there is no $x' \in X$ such that for each $i \in N$, $x'_i \; R_i \; x_i$, and there is $i \in N$ for which $x'_i \; P_i \; x_i\}$.

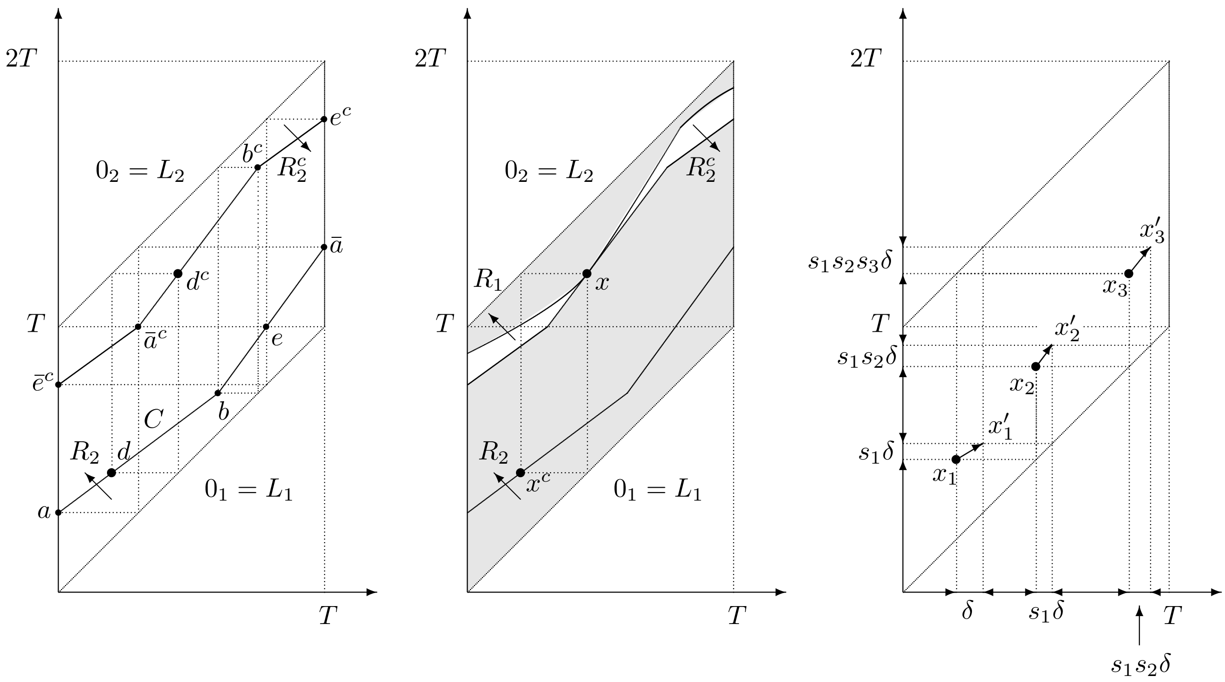
We develop a method to easily identify efficient partitions in the two-agent case. This method will also help us provide answers to a number of other questions. Let $N \equiv \{1, 2\}$ and $R \equiv (R_1, R_2) \in \mathcal{R}^N$. Let $x \equiv (x_1, x_2) \in X$. To find out whether $x \in P(R)$, we determine the set of partitions that are at least as desirable as x for each agent. Once an agent's arc is chosen, the other agent's arc is known, so in arc space a partition can be represented by a single point. To take advantage of this fact, we need to represent agent 2's induced preferences over agent 1's arcs. Figure 4a illustrates the construction. It is akin to that underlying the Edgeworth box. The counterpart of the symmetry operation with respect to the center of the box is a **complement operation**.

[6]Given the monotonicity assumption we have made on preferences, the apparently weaker notion of efficiency according to which a partition is disqualified only if there exists some other partition that all agents prefer—in other contexts, it is known as the "weak Pareto solution"—is equivalent to the definition we use.

Given each arc $a \in A$, it consists in constructing a rectangle as follows. One vertex is a. From a, one draws horizontal and vertical lines to L_1 and L_2, to the right and up if a is below the horizontal line of ordinate T, and to the left and down if a is above that line, (or both if a is on the line). The points where L_1 and L_2 are reached are two additional vertices of the rectangle, which can then be completed. The fourth vertex is the complement of a. If $a \in L_1 \cup L_2$, the rectangle reduces to a vertical segment. (The partition at which one agent receives nothing and the other receives everything is a special case: it is represented by any pair consisting of a point of L_1 together with any point of L_2.) Let us consider an indifference curve for agent 2. An example is $C \equiv \text{seg}[a, b] \cup \text{seg}[b, \bar{a}]$ in Figure 4a. If agent 2 receives arc d on C, agent 1 receives the complement arc d^c obtained by the rectangle construction just described. The locus of d^c as d describes C is a "complement" indifference curve for agent 2. Let R_2^c denote the map generated by the complement of C as C varies. For that map, direction of increasing satisfaction is southeast.

In what follows, when $|N| = 2$, we use our four-sided diagrams in two ways depending upon what is most convenient. Either we measure all arcs from the same origin, as we have done until now, or we subject agent 2's preferences to the transformation explained above. Then, the partition at which agent 1 consumes the empty arc and agent 2 consumes the entire circle is represented by any point of L_1 and the complement partition by any point of L_2. Both L_1 and L_2 are always part of the Pareto set. They are "origins", and to emphasize this fact, we also use the notation 0_1 and 0_2 then. On the figures, the appearance of this notation signals our using agent 2's complement map. The complement operation preserves tangency and piecewise linearity (Figure 4a).[7] It inverts slopes. This can be seen by a simple calculation. For instance, consider the two arcs d and b in Figure 4a, which belong to a linear part of an indifference curve. Calculate the coordinates of their complements d^c and b^c, and then the slopes of $\text{seg}[d, b]$ and $\text{seg}[d^c, b^c]$. These slopes are inverse of each other. A concave curve below the line of ordinate T is mapped into a convex curve above that line (for instance the convex broken line segment $\text{seg}[d, b] \cup \text{seg}[b, e]$ is mapped into the broken concave line segment $\text{seg}[d^c, b^c] \cup \text{seg}[b^c, e^c]$), and the opposite holds for a convex curve; it is mapped into a concave curve. So the complement operation does not preserve convexity.

[7] Recall that convexity of upper contour sets in our representation is not preserved under changes of origin.



(a) (b) (c)

Figure 4: Pareto efficient partitions. (a) Calculating the "complement" of a typical indifference curve of agent 2. (b) Efficiency for $|N| = 2$. By maximizing agent 1's preferences over the complement of a typical indifference curve of agent 2's, we obtain an efficient partition. (c) Efficiency for general $|N|$.

Now, using the representation just described, Pareto efficiency of $x \in X$ for (R_1, R_2) means as usual that agents 1 and 2's upper contour sets at x are separated. In the example of Figure 4b, they are, so x is efficient. When the same origin is used for both agents, and when preferences are smooth, separation takes the form of tangency, and tangency translates into an inverse relationship of the slopes of their indifference curves at x_1 and x_2, because, as we noted above, the complement operation inverts slopes. This inverse relationship of slopes, which is a necessary condition for efficiency, is of course not sufficient, again because of the lack of convexity of preferences for the model: two indifference curves that are tangent at one point can cross further away. Another implication of the lack of convexity is that the Pareto set may have several components, and we will see an example later.

In the n-agent case, we have the following necessary condition for efficiency:

Theorem 1 *A necessary condition for efficiency for a profile of smooth pref-*

erences is that the product of the slopes of the agents' indifference curves at their respective consumptions be equal to 1.

Proof: Let $R \in \mathcal{R}^N$ and $x \in P(R)$. For each $i \in N$, let $s_i(x_i)$ be the slope of agent i's indifference curve passing through x_i at that point (Figure 4c). For simplicity and without loss of generality, let $N \equiv \{1, 2, \ldots, n\}$, and suppose that at x, agents consume in the order $1, 2, \ldots, n$, and that agent 1 is the one whose starting point is the closest to 0. A small delay of δ in agent 1's starting point should be accompanied by a delay of $s_1(x_1)\delta$ in his endpoint to keep him on the same indifference curve. This is the delay in agent 2's starting point. This delay should be accompanied by a delay of $s_2(x_2)s_1(x_1)\delta$ in his endpoint to keep him on the same indifference curve. This in turn determines the delay in agent 3's endpoint, and so on. This goes on until agent n: his endpoint should be delayed by $s_n(x_n)s_{n-1}(x_{n-1}) \ldots s_1(x_1)\delta$. This delay is of course also the delay in agent 1's starting point, namely δ. Thus, $s_n(x_n)s_{n-1}(x_{n-1}) \ldots s_1(x_1)\delta = \delta$, or more compactly $\Pi s_i(x_i) = 1$. □

6 Fairness

Next, we turn to fairness requirements. First, we consider allocations at which each agent finds what he receives at least as desirable as what any other agent receives (Foley, 1967):

No-envy solution, *F*: For each $R \in \mathcal{R}^N$, $F(R) \equiv \{x \in X$: for each $\{i, j\} \subseteq N$, $x_i \; R_i \; x_j.\}$

Our first result here is an existence result for envy-free partitions. The standard way of proving the existence of such allocations is to appeal to equal division. Because of the non-homogeneity of the dividend, there is no meaning to equal division here, and some other argument must be found.[8]

Theorem 2 *For each problem, the set of envy-free partitions is non-empty.*

First, we give an informal argument in the two-person case. In fact, we prove a slightly stronger result: for each $t \in [0, T]$, there is an envy-free partition at which one agent receives an arc that begins at t. A simple proof

[8]The circle can certainly be divided into equal parts, but these parts would in general be viewed differently by different agents, and no-envy would not be guaranteed.

is obtained by applying the well-known "divide-and-choose" protocol. Let $N \equiv \{1, 2\}$. We partition the circle into two arcs, one of which starts at t (and the other ends at t), that agent 1, say, finds indifferent to each other. This is possible, by continuity of preferences. In Figure 5a, the point b is too low on seg$[(t,t),(t,t+T)]$, the set of arcs beginning at t, because agent 1's indifference curve through b passes below b^c. The opposite holds for d, which is too high. But agent 1's indifference curve through α passes through α^c. (By monotonicity of his preferences, there is no other arc β beginning at t for which $\beta\ I_1\ \beta^c$.) Then, we give to agent 2 the arc in $\{\alpha, \alpha^c\}$ that he prefers, or either arc if he is indifferent between them, and to agent 1 the other arc. The partition so obtained is envy-free. We now turn to the case of more than two agents.[9]

Proof: It is a corollary of the theorem stating the existence of envy-free allocations in the linear model (Stromquist, 1980; Su, 1999). Indeed, let $t \in [0, T]$ and let us consider the partitions having one component that begins at t (and therefore one component that ends at t). By that theorem, at least one of these partitions is envy-free. □

Next, we turn to the problem of identifying the entire set of envy-free allocations in the two-person case. It is most conveniently solved by introducing the notion of an **envy boundary** for an agent (an extension of a concept proposed by Kolm, 1972, for the classical model). An arc belongs to an agent's envy boundary if the agent finds it indifferent to its complement. To identify such points, for each indifference curve C, we construct the complement of C, and we take the intersection of C with its complement. The locus of the points of intersection is what we are looking for. (Of course, if an indifference curve is too low or too high, it does not intersect its complement.) The locus is an upward-sloping curve from seg$[(0,0),(0,T)]$ to seg$[(0,T),(T,T)]$ that is globally invariant under the complement operation since it consists of points that are complement of each other. Conversely, given any curve with these properties, there is a preference relation for which it is the envy boundary. As proved above, for each $t \in [0, T]$, there is $x_1 \in A$ such that $b(x_1) = t$ and $x_1\ I_1\ x_1^c$.

[9] Barbanel and Brams (2004) note that a "moving-knife" procedure that they develop for the linear case can be used to produce an envy-free partition in the four-agent circular case at which at most one agent receives a disconnected union of (at most two) intervals. They do not address the efficiency issue.

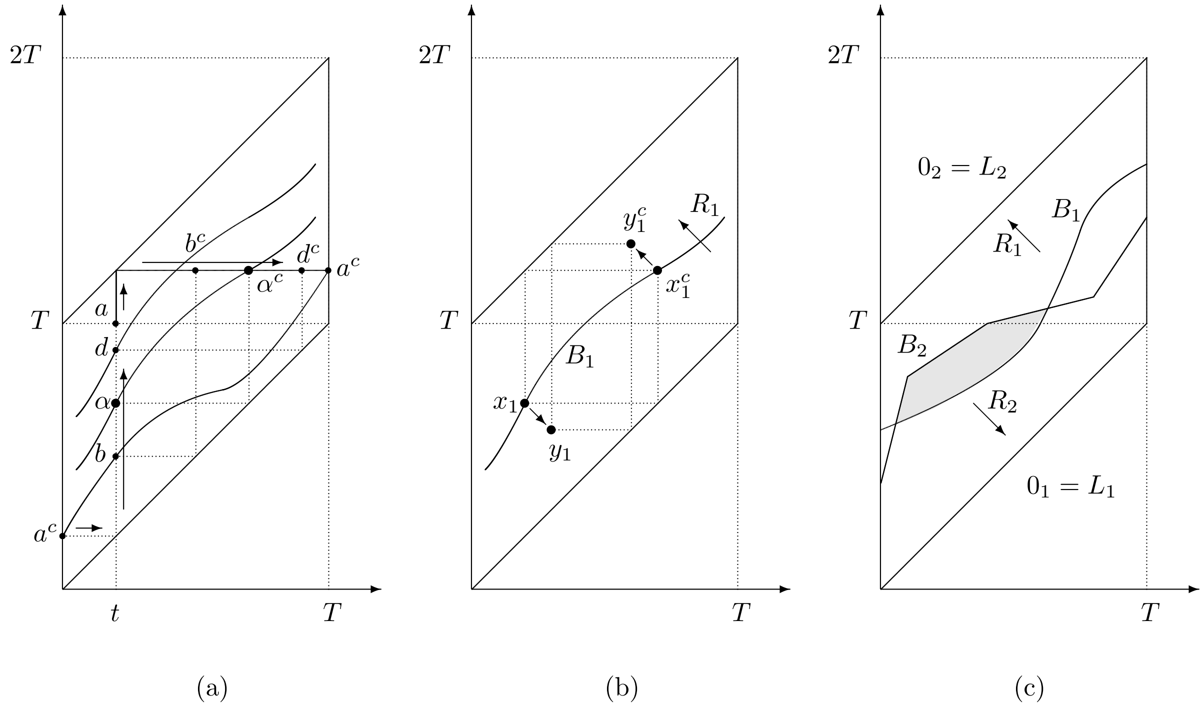


Figure 5: Envy-free partitions. (a) Identifying the arc on agent 1's envy boundary that begins at a particular point t. (b) For agent 1 not to envy agent 2, he should receive an arc above his envy boundary. (c) For a partition to be envy-free, it should be above agent 1's envy boundary, B_1, and below the complement of agent 2's boundary, B_2.

Remark 1. The envy boundary associated with an additive preference relation is an indifference curve of the map. The reason is that such a map is invariant under the complement operation. This invariance is shown in Figure 6a for a partition of $[0, T]$ into two homogeneous components and in Figure 6b for a partition into three homogeneous components. As already noted when we first discussed the two-component case, in rectangles T^1 and T^2 of Figure 6a, the slopes of indifference curves are inverse of each other. Similarly, in the three-component case, the rectangles come in pairs in which slopes are the inverse of each other. Suppose that the circle is partitioned into three regions, $A^1 \equiv [0, t^1]$, $A^2 \equiv [t^1, t^2]$, and $A^3 \equiv [t^2, T]$, with each arc in A^2 being twice as valuable as each arc of the same length in A^1, and each arc in A^3 being twice as valuable as each arc of the same length in A^2. Then, in rectangles T^1 and T^2, indifference curves are segments of slopes $\frac{1}{2}$ and 2 respectively; in rectangles T^5 and T^6, they are segments of $\frac{1}{2}$ and 2 respectively; in rectangles T^3 and T^4, they are segments of slopes $\frac{1}{4}$ and 4 respectively.

Next, we claim that for agent 1 not to envy agent 2 at y, it is necessary and sufficient that y_1 be on or above his (agent 1's) envy boundary, B_1. The proof is illustrated on Figure 5b. The point y_1 is below B_1. Thus,

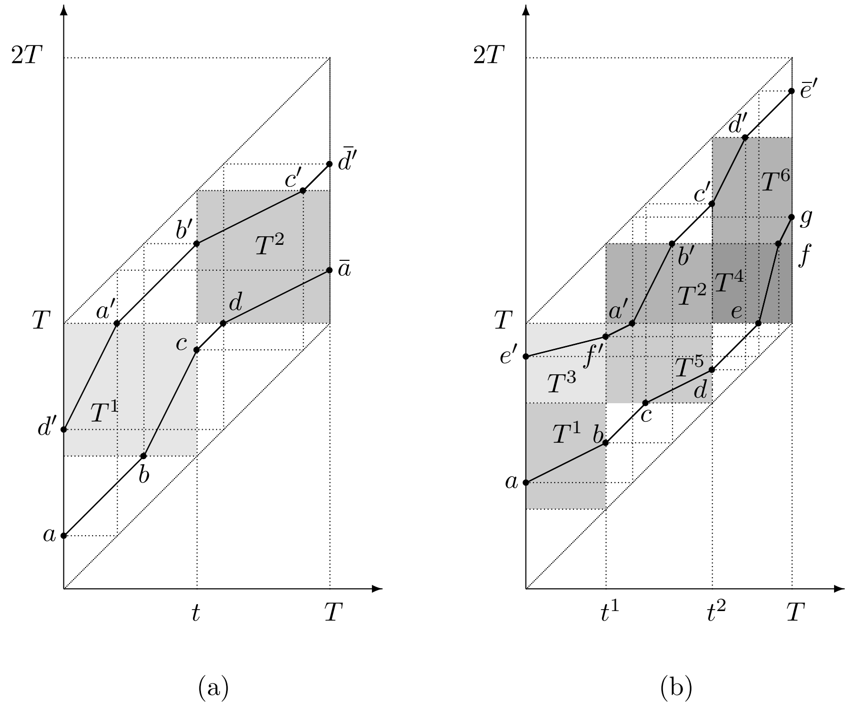


Figure 6: Calculating complements of two additive relations. Relations corresponding to (a) a two-component division of the circle, and to (b) a three-component division.

there is $x_1 \in B_1$ to the northwest of y_1. Since $x_1 \in B_1$, then $x_1 \; I_1 \; x_1^c$. By monotonicity of preferences, $x_1 \; P_1 \; y_1$. Since x_1 is to the northwest of y_1, y_1^c is to the northwest of x_1^c. By monotonicity of preferences, once again, $y_1^c \; P_1 \; x_1^c$. Altogether, $y_1^c \; P_1 \; x_1$. Conversely, and by the same argument, if y_1 is above B_1, agent 1 does not envy agent 2 at y.

Finally, and using agent 2's complement preferences, the entire set of envy-free allocations is obtained as the intersection of the area above B_1 and the area below the complement of B_2, which is B_2 itself because of the invariance property that B_2 satisfies (Figure 5c).

For the interval division problem, a number of remarkable facts have been noted. First, say that a solution satisfies **Pareto indifference** if whenever it selects a partition, it also selects any partition that all agents find indifferent to it. For interval division, the no-envy solution satisfies *Pareto indifference* (Thomson, 1987). This is not the case for the "classical" problem of fair division, which has to do with the division of an infinitely divisible and homogeneous good among agents with continuous, monotonic, and convex preferences (Thomson, 1987). Also, for the interval division problem, an envy-free partition is necessarily efficient (Berliant, Dunz, and Thomson, 1992). This is of course not true for the classical model. Finally, envy-free allocations exist. For the classical model, under standard assumptions on preferences, envy-free and efficient allocations exist, but they may not if

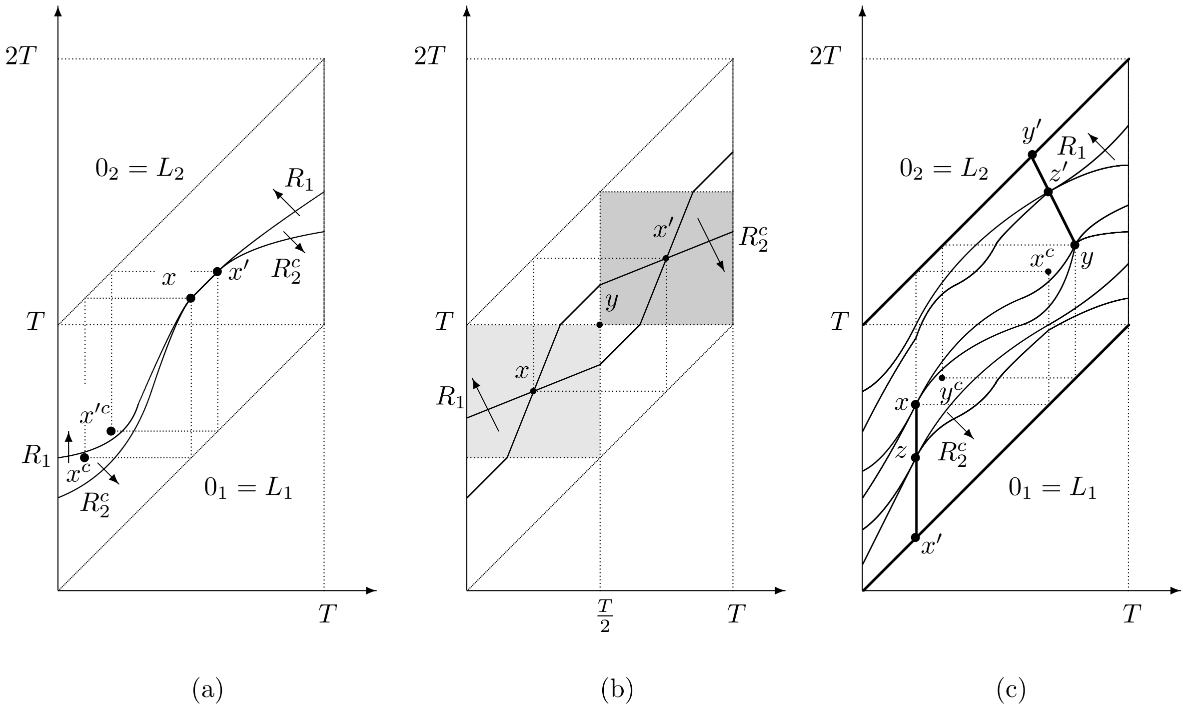


Figure 7: Relating no-envy and efficiency (Theorem 3). Panels (a), (b), (c) illustrate parts (a), (b), and (c) of the theorem.

preferences are not convex (Varian, 1974). Here, we have the following:

Theorem 3 (a) *The no-envy solution violates* Pareto-indifference*.*

(b) *There are cake division problems admitting partitions that are envy-free but not efficient.*

(c) *There are cake division problems with no efficient and envy-free partition.*

Proof: (a) The proof is by means of an example, illustrated in Figure 7a. We do not give analytical expressions for preferences as nothing would be gained from the formulas. (The examples used to prove the other parts of the theorem are also specified geometrically only.) The partition x is envy-free for the profile $R \in \mathcal{R}^N$ represented there. The partition x' is Pareto-indifferent to it. However, agent 1 envies agent 2 at x'. Also, x happens to be efficient for R. Thus, the same example shows that the no-envy and Pareto solution violates *Pareto indifference*.

(b) The proof is by means of an example, illustrated in Figure 7b. The

partition x is envy-free for the profile $R \in \mathcal{R}^N$ represented there. However, x is Pareto-dominated by the partition y.
(c) The proof is by means of an example, illustrated in Figure 7c. (The inspiration is Varian, 1974.) The Pareto set of the profile $R \in \mathcal{R}^N$ represented there has two components. One is $L_1 \cup \text{seg}[x', x]$; the other is $\text{seg}[y, y'] \cup L_2$. The partitions x and y are Pareto-indifferent. At any point $z \in \text{seg}[x', x]$, the two indifference curves are tangent, and as z gets closer and closer to x, the two indifference curves passing through z get closer and closer to each other around y. The point y is a second point of contact of the two curves passing through x. However, two indifference curves tangent at a point $z \in \text{seg}[y, y']$ do not make contact at another point around x any more. At x, agent 1 envies agent 2, and by monotonicity of preferences, so does he at any other point of the first component of $P(R)$. Similarly, at y, agent 2 envies agent 1, and so does he at any other point of the second component. □

The proofs of Parts (a) and (c) of Theorem 3 rely on the admissibility of all continuous and monotone preferences. For the special case of additive preferences, the situation looks better. Concerning (a), at least in the two-agent case, the no-envy and Pareto solution does satisfy *Pareto indifference*. Part (b) however is still valid. Indeed, the example of panel (b) represents a problem with additive preferences as previously illustrated in Figure 2 (the circle is divided into two homogeneous areas of equal sizes; agent 1 prefers the bottom part and agent 2 the top part; the symmetry of their preferences with respect to y in panel (b) reveals that their relative degree of preferences for the two parts is the same). However, there is no counterpart of Part (c) in the two-agent case. In fact, we have the following existence result then:

Theorem 4 *In a two-agent problem in which at least one agent has additive preferences, envy-free and efficient partitions exist.*

Proof: To fix the ideas, suppose that agent 1 has additive preferences. As indicated in Remark 1, his envy boundary is one of his indifference curves. Thus, the partition obtained by maximizing agent 2's preference relation over the complement of agent 1's indifference curve is efficient. It is also envy-free since an envy boundary is composed of points that are the complement of each other. □

Theorem 4 is a slight generalization of an existence result for economies with two agents both of whom have additive preferences (Barbanel and

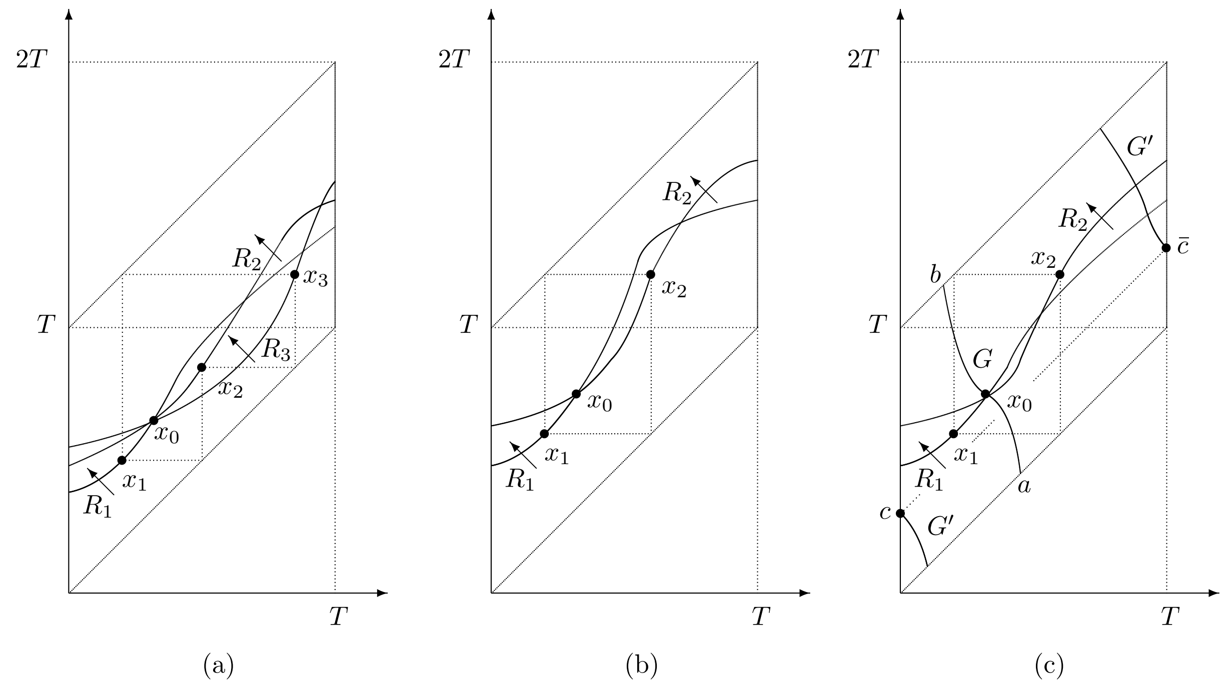


Figure 8: Egalitarian-equivalence. (a) Here, $N \equiv \{1,2,3\}$. Partition $x \in E(R)$ with reference arc x_0. (b) For $|N| = 2$, if $x \in F(R)$, then $x \in E(R)$. (c) Selection from E obtained by requiring the reference bundle to lie on $G \in \mathbb{R}^2_+$.

Brams, 2005). For more than two agents, Chambers (2004) and these authors settle in the negative the question of existence of envy-free and efficient allocations. (The first authors' proof is constructive.) Both of these contributions involve preferences representable by measures that are not mutually absolutely continuous however. The three-agent case is still unsettled.

The second most important distributional requirement in the fairness literature is arguably that there should exist a "reference" consumption that each agent finds indifferent to his own consumption (Pazner and Schmeidler, 1978). Here, it takes the following form:

Egalitarian-equivalence solution, *E*: For each $R \in \mathcal{R}^N$, $E(R) \equiv \{x \in X$: there is $x_0 \in A$ such that for each $i \in N$, $x_i \; I_i \; x_0$.$\}$

The definition is illustrated in Figure 8a. It is easy to see that in the two-person case, if a partition is envy-free, it is egalitarian-equivalent. Indeed, if agent 1's indifference curve through his arc passes above agent 2's arc and agent 2's indifference curve through his arc passes above agent 1's arc, the two curves necessarily intersect. Any point of intersection can serve as reference arc to rationalize the partition as egalitarian-equivalent. This implication does not hold for more than two agents, as is easily shown. Equally straightforward is that an egalitarian-equivalent and efficient allocation need not be envy-free. As far as existence is concerned, we have the following positive result:

Theorem 5 *The set of egalitarian-equivalent and efficient partitions of each problem is non-empty.*

Proof: The proof is standard. Let G be a continuous path from some point $a \in L_1$ to some point $b \in L_2$ that is weakly monotone in the northwesternly direction. Non-degenerate horizontal or vertical segments are permitted. Two examples of such paths are illustrated in Figure 8c, denoted G and G'. The path G', which has two components, does not appear to be monotone, but it is once one remembers that its endpoints, c and $\bar{c}$ on the figure, represent the same partition. Let $R \in \mathcal{R}^N$. For each $i \in N$, let u_i be the numerical representation of R_i obtained by assigning to each arc x_i a value equal to the length of the part of G that lies between a and the point that is indifferent to x_i (given our assumptions on preferences, this point is uniquely defined, up to inessential duplication on the vertical lines of abscissa 0 and T).[10] Using these representations, we take the image in $\mathbb{R}^N_+$ of the set of feasible partitions. This point exists as the latter is a compact set. Then, in that space, we identify the maximal feasible point of equal coordinates. The partition(s) whose image(s) is (are) this point is (are) egalitarian-equivalent and efficient. □

We close this section by discussing the version of the model in which each agent is endowed with an arc, the endowment profile defining a partition. For this version, two natural solutions can be defined. The **endowment lower bound** solution selects all the partitions that each agent finds at least as desirable as his endowment.[11] This solution is of course not empty since it contains the endowment profile. That is also the case for its intersection with the Pareto solution. The **core** generalizes the idea to groups of agents. No group should be able to make all of its members at least as well off, and at least one of them better off, by redistributing among themselves the resources they control. Here, we have the following negative result:

[10] In the case of a broken path, such as G' in the figure, if the indifference through x_i happens to cross the path in the second component of the path, one uses the sum of the length of the component of G' containing c and the length of the part of the second component from $\bar{c}$ to the point of G' that is indifferent to x_i.

[11] A common expression for this requirement is "individual rationality".

Theorem 6 *The core of a problem with endowments may be empty.*

Proof: Let $N \equiv \{1,2,3\}$ and $[0,T]$ be divided into three regions, $A^1 \equiv [0, \frac{T}{3}]$, $A^2 \equiv [\frac{T}{3}, \frac{2T}{3}]$, and $A^3 \equiv [\frac{2T}{3}, T]$, such that for each $i \in N$, the value of each arc x_0 be determined as follows: let $x_0^1 \equiv x_0 \cap A^1$, $x_0^2 \equiv x_0 \cap A^2$, and $x_0^3 \equiv x_0 \cap A^3$. Let $\ell(x_0)$ denote the length of arc x_0. Then, for each $i \in N$, let $u_i(x_i) \equiv \ell(x_i^1)\ell(x_i^2)\ell(x_i^3)$. Also, let $\omega_1 \equiv [\frac{T}{6}, \frac{3T}{6}]$, $\omega_2 \equiv [\frac{3T}{6}, \frac{5T}{6}]$, and $\omega_3 \equiv [\frac{5T}{6}, \frac{7T}{6}]$. For an agent to reach a positive utility, he should consume some positive amount of each region. This is possible only if he has access to at least one of the three regions in its entirety. Thus, an agent on his own can only reach a utility of 0. When joined by another agent, only one or the other, but not both together, can reach a positive utility. Thus, the union of the resources they control should go in its entirety to only one of them, resulting in the vectors $(\frac{T^3}{6\cdot3\cdot6}, 0)$ and $(0, \frac{T^3}{6\cdot3\cdot6})$. The non-negative vectors obtained from those by disposing of utility are also attainable. By the same reasoning, in the grand coalition, only one of the three agents can reach a positive utility. Here too, the union of the resources they control, namely the entire circle, should go to only one of them. Thus, the grand coalition can reach the vectors $(\frac{T^3}{3\cdot3\cdot3}, 0, 0)$, $(0, \frac{T^3}{3\cdot3\cdot3}, 0)$, and $(0, 0, \frac{T^3}{3\cdot3\cdot3})$. The non-negative vectors obtained from them by disposing of utility are also attainable. The first vector is blocked by $\{2,3\}$, the second by $\{1,3\}$, and the third by $\{1,2\}$. All other vectors are blocked. Thus, the core is empty. □

7 Solidarity and incentives

In this section, we turn to several requirements of solidarity, and we consider strategic issues.

The general idea of solidarity is that, when the environment in which some group of agents find themselves change, and if no one in particular is responsible for the change—no one deserves any credit for it if it permits a Pareto improvement, and no one is to blame if the initial welfare profile is not feasible anymore—the welfares of all of them should be affected in the same direction. Different specific properties can be obtained by considering particular changes in the environment. **Welfare-domination under preference-replacement** (Moulin, 1987; see Thomson, 1999, for a survey of the literature concerning this property), says that as the preferences of an agent change, the welfares of all other agents should be affected in the same direction. Second, **population monotonicity** (Thomson, 1983; see Thomson, 1995, for a survey of the literature concerning this property) says that as the population of agents increases, all agents initially present should end up at most as well off as they were initially.[12]

In the process of proving Theorem 5, we have defined *single-valued* selections from the egalitarian-equivalence and Pareto solution that satisfy **welfare-domination under preference-replacement** and **population monotonicity**. Let $\boldsymbol{\mathcal{G}}$ be the family of paths satisfying the properties listed there, and for each $G \in \mathcal{G}^N$, let $\boldsymbol{E^G}$ be the corresponding solution.

Theorem 7 *For each $G \in \mathcal{G}$, the rule E^G satisfies welfare domination under preference-replacement and population monotonicity.*

Proof: Let $R \in \mathcal{R}^N$ and $x \in E^G(R)$, with associated reference bundle x_0. Let $i \in N$ and $R'_i \in \mathcal{R}$. Let $y \in E^G(R'_i, R_{-i})$, with associated reference bundle y_0. If y_0 is higher than x_0 on G, then all $j \in N \setminus \{i\}$ end up at least as well off as they were initially; otherwise they all end up at most as well off as they were initially. Thus, the rule satisfies the first property.

The proof that it satisfies the second property is similar and we omit it. □

A more general family of rules enjoying the same properties is defined as follows. Let $D\colon [0,1] \to 2^X$ be a continuous function such that $D(0) = L_1$ and $D(1) = L_2$. Then, for each $i \in N$ and each $x_i \in A$, let $u_i(x_i)$ be equal to the maximal $\lambda \in \mathbb{R}_+$ such that the maximizer of R_i on $D(\lambda)$ be indifferent to x_i. Finally, select the partition(s) such that, using these welfare indices, all agents reach equal welfares.

An interesting special case is when the choice sets are segments of slope 1 from the vertical axis to the line of abscissa T. Each such segment represents

[12] Another property that has been important in the development of the axiomatic program is that as opportunities enlarge, all agents should benefit. It is usually referred to as *resource monotonicity.* In the problem as we formulated it, it is not possible to enlarge the feasible set however. One can certainly imagine cutting open a circle so as to insert an arc. New arcs will become available. However, some existing arcs will disappear. The feasible set is a superset of the old one but the points of the triangle with vertices $(0, T)$, (T, T), and $(T, 2T)$ represent different consumptions and therefore the shape of indifference curves in that triangle will change. Although it is not legitimate to require that an enlargement of the circle should benefit everyone, it is meaningful to require instead that the welfares of all agents should be affected in the same direction.

all the arcs of a given length.[13]

We conclude with an examination of strategic issues. For that purpose, we first consider the version of the model in which each agent is initially endowed with some arc, the resulting endowment profile defining a partition. We look for selections from the intersection of the endowments lower bound solution, with the Pareto solution satisfying the following strong requirement of non-manipulability. A rule is **strategy-proof** if no agent ever has an incentive to misrepresent his preferences: formally, for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, we have $\varphi_i(R)\ R_i\ \varphi_i(R'_i, R_{-i})$. Not surprisingly, given the generality with which such results have been proved in the literature, we have an impossibility, which is a counterpart for the present model of a fundamental result due to Hurwicz (1978): there is no *strategy-proof* selection from the endowments lower bound and Pareto solution.

A stronger result is available however. Say that a rule is **dictatorial** if there is an agent, chosen once and for all, who always receives his most preferred consumption, here the entire circle. The following result is a counterpart for our model of the Gibbard (1973) and Satterthwaite (1975) theorem, which pertains to the abstract Arrovian model of social choice. A number of versions of the theorem have now been established for concretely specified resource allocation models, Zhou (1991) and Schummer (1997) being the most relevant to our analysis here. The proof strategy is as follows. We begin with a special subdomain of the additive domain. The first step is a simple observation concerning the limited possible shapes of the Pareto sets for preference profiles in that subdomain. The second step is an invariance property of *strategy-proof* rules under Maskin-monotonic transformation of preferences.[14] The third step derives the dictatorship conclusion on the special subdomain. The fourth step is a "contamination argument": it extends the dictatorial conclusion to the entire domain.

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[13]This is an application of the notion of equal-opportunity equivalence proposed by Thomson (1994).

[14]A Maskin-monotonic transformation of a preference relation at a point involves enlarging the lower contour of the initial relation at that point (Maskin, 1999). For a "strict" Maskin-monotonic transformation, the old and the new lower contour sets at that point should only have that point in common. On the additive domain, there is no strict Maskin-monotonic transformation. Thus, the invariance conclusion obtained for such transformations that is a common element of proofs on *strategy-proofness* has no exact counterpart. Instead, the conclusion is that the choice should remain on a line of slope 1. It is enough for the argument to proceed to the dictatorial conclusion, as we show.

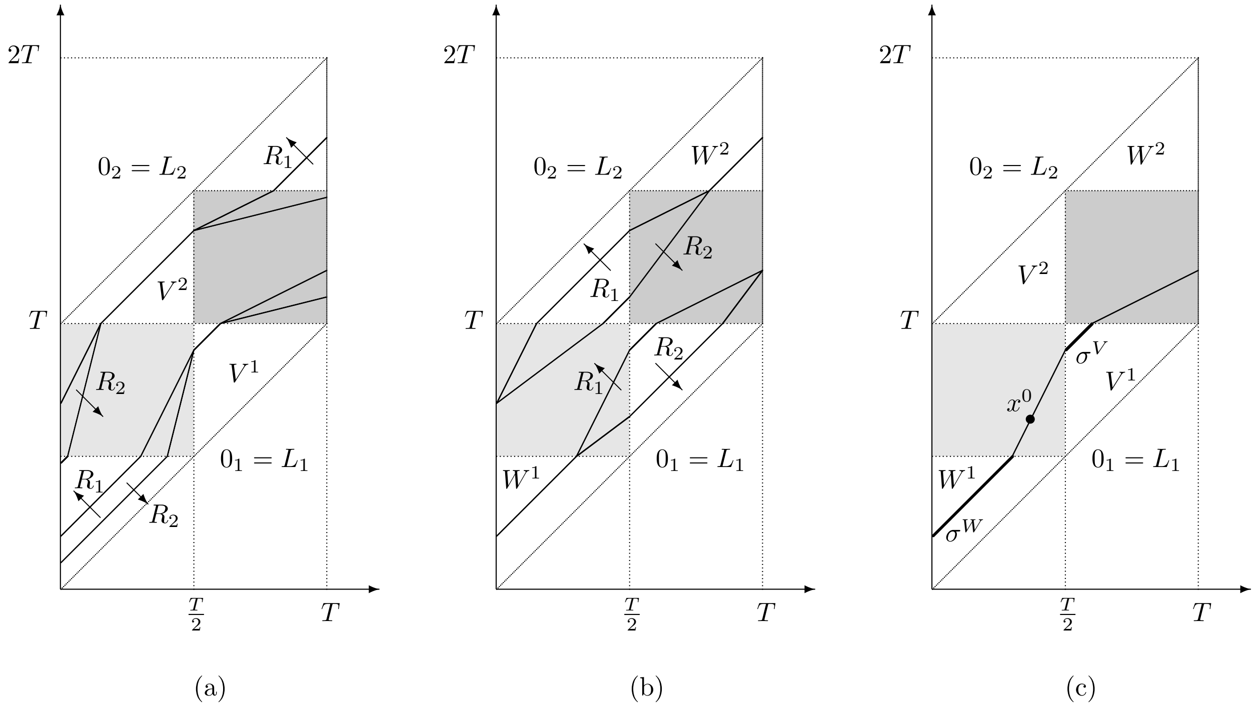

Figure 9: Strategy-proofness for additive preferences. Step 1 of Theorem 8. There are three possible shapes for the Pareto set.

Theorem 8 *For $|N| = 2$. A selection from the Pareto solution is strategy-proof if and only if it is dictatorial.*

Proof: We omit the easy proof that the dictatorial rules satisfy the axioms named in the theorem. Conversely, let $\varphi \in P$ be a *strategy-proof* rule. We first consider its behavior on a subdomain of economies with additive preferences, as defined in Section 4. Let $A^1 \equiv [0, \frac{T}{2}]$, and $A^2 \equiv [\frac{T}{2}, T]$. Let $\ell(x_0)$ denote the length of x_0. For each $i \in N$ and each $x_i \in A$, let $k_i > 0$, and let R_i be the preference relation represented by the function $u_i \colon A \to \mathbb{R}_+$ defined by $u_i(x_i) \equiv \ell(x_i \cap A^1) + k_i \ell(x_i \cap A^2)$. Let $\mathcal{R}^*_{add}$ be the class of these preferences. The vertical line of abscissa $\frac{T}{2}$ and the horizontal lines of ordinates $\frac{T}{2}$ and $\frac{3T}{2}$ partition arc space into six regions, labelled in Figure 9. They are the triangles V^1 and V^2, the triangles W^1 and W^2, and two squares (unlabelled but shaded).

Step 1: Identifying the only three possible shapes of the Pareto set. For each $R \in \mathcal{R}^{*N}_{add}$, $P(R)$ has one of three shapes:

$$P(R) = \begin{cases} V \equiv L_1 \cup V^1 \cup V^2 \cup L_2 & \text{if } k_1 > k_2; \\ W \equiv L_1 \cup W^1 \cup \text{seg}[(T,T),(T,\frac{3T}{2})] \cup W^2 \cup \text{seg}[(0,\frac{T}{2}),(0,T)] \cup L_2 & \text{if } k_1 < k_2; \\ X & \text{otherwise.} \end{cases}$$

We omit the easy proof and only refer to Figure 9a for an illustration.

Note that any two partitions on a segment in V^1 of slope 1 are Pareto-indifferent. Indeed, both agents' indifference curves are segments of slope 1 in these regions. The same comment applies to any two partitions on segments of slope 1 in V^2, in W^1, in W^2, and of course in L_1 and in L_2. From now on, when we speak of a "segment in V^1" (say), we mean the maximal segment of slope 1 in V^1.

Step 2: For each pair $\{R, R'\} \subset \mathcal{R}^{*N}_{add}$ such that $P(R) = P(R') = V$, φ makes the same choice up to Pareto-indifference. Similarly, for each pair $\{R, R'\} \subset \mathcal{R}^{*N}_{add}$ such that $P(R) = P(R') = W$, φ makes the same choice up to Pareto-indifference. To prove this, let $\{R, R'\}$ be as specified in the first case of the hypothesis, such that $k_1 > k_2$ and $k'_1 > k'_2$ (The second case is analyzed in a similar way).

Case 1: $k'_1 \geq k_1 \geq k_2 \geq k'_2$ or $k_1 \geq k'_1 \geq k'_2 \geq k_2$. (Then, R is obtained from R' by a Maskin-monotonic transformation at $\varphi(R)$ in the first subcase, and the reverse is true in the second subcase.) Without loss of generality, we assume the first subcase. Suppose that agent 1's true preferences are R_1 and that agent 2 announces R_2. By *strategy-proofness*, agent 1 does not benefit from switching to R'_1. Thus $\varphi(R'_1, R_2)$ belongs to a segment in V that lies to the southeast of the segment containing $\varphi(R)$. Similarly, suppose that agent 1's true preferences are R'_1. By *strategy-proofness*, he does not benefit from switching to R_1. Thus, $\varphi(R)$ belongs to a segment in V that lies to the northwest of the segment containing $\varphi(R'_1, R_2)$. These two conclusions can be met together only if $\varphi(R)$ and $\varphi(R'_1, R_2)$ belong to the same segment in V.

Case 2: $k'_1 \geq k_1 \geq k'_2 \geq k_2$ or $k_1 \geq k'_2 \geq k'_1 \geq k_2$. The profiles R and (R'_1, R_2) are related as in Case 1, and so are the profiles (R'_1, R_2) and R'. The conclusion established for Case 1, applied twice, gives us that $\varphi(R)$ and $\varphi(R')$ belong to the same segment in V.

Let σ^V be the segment in V to which, for each $R \in \mathcal{R}^{*N}_{add}$ such that $P(R) = V$, the partition selected by φ belongs, and σ^W be the segment in W to which, for each $R \in \mathcal{R}^{*N}_{add}$ such that $P(R) = W$, the partition selected by φ belongs.

Step 3: Either agent 1 dictates on $\mathcal{R}^*_{add}$, or agent 2 dictates on $\mathcal{R}^*_{add}$. We first show that either $\sigma^V, \sigma^W \subset L_1$ or $\sigma^V, \sigma^W \subset L_2$. For that purpose, let $R_0 \in \mathcal{R}^{*N}_{add}$ and $x^0 \equiv \varphi(R_0, R_0)$. There is $R'_1 \in \mathcal{R}^*_{add}$ such that $P(R'_1, R_0) = V$ and there is $R''_1 \in \mathcal{R}^*_{add}$ such that $P(R''_1, R_0) = W$.

Suppose that agent 1's true preferences are R_0 and that agent 2 announces R_0. For agent 1 not to benefit by switching to R_1', σ^V should lie below his R_0-indifference curve through x_0, and for him not to benefit by switching to R_1'', σ^W should lie below that same indifference curve. Agent 2 also has the power to make the Pareto set appear to be either V or W. When his true preferences are R_0 and agent 1 announces R_0, for him (agent 2) not to benefit from such misrepresentation, two parallel statements should hold. These four conclusions hold together only if σ^V and σ^W are both subsets of the curve that is agent 1's R_0-indifference curve through x^0 as well as agent 2's R_0-indifference curve through that point. Now, let $R_0' \neq R_0$. The conclusion just reached for (R_0, R_0) also applies to (R_0', R_0'). This is possible only if $\sigma^V, \sigma^W \subset L_1$ or $\sigma^V, \sigma^W \subset L_2$, so in fact, $\varphi(R) \in L_1$ for each $R \in \mathcal{R}_{add}^{*N}$ or $\varphi(R) \in L_2$ for each $R \in \mathcal{R}_{add}^{*N}$.

Suppose that the first case holds, namely $\sigma^V, \sigma^W \subset L_1$. Returning to the profile (R_0, R_0), it is now clear that then, $\varphi(R_0, R_0) \in L_1$. Altogether, agent 2 dictates on $\mathcal{R}_{add}^{*N}$. If the other case holds, then agent 2 dictates on $\mathcal{R}_{add}^{*N}$.

Step 4: For each $i \in N$, if agent i dictates on $\mathcal{R}_{add}^{*N}$, he dictates on the entire domain. Suppose without loss of generality that agent 1 dictates on $\mathcal{R}_{add}^{*N}$. Let $(R_1, R_2) \in \mathcal{R} \times \mathcal{R}_{add}^{*}$. By switching to $R_1' \in \mathcal{R}_{add}^{*}$, agent 1 can ensure that he receives the entire circle. For him not to benefit from such misrepresentation, he should also receive the entire circle when his true preferences are R_1. So, he dictates on $\mathcal{R} \times \mathcal{R}_{add}^{*}$. Now, let $R \in \mathcal{R}^N$. If $R_2 \in \mathcal{R}_{add}^{*}$, agent 2 receives the empty arc. He should not be able to reach a non-empty arc by switching to $R_2' \in \mathcal{R} \setminus \mathcal{R}_{add}^{*}$, so once again, at R, agent 1 should receive the entire circle. □

Non-efficient and *strategy-proof* rules that are not dictatorial can be defined as follows. Select a subset of arc space. Agents announce preferences. Agent 1 is given the arc in that subset that he prefers according to his announced preferences. In case of multiple maximizers, a tie-breaker is also specified. Agent 2 receives the complement. Such a rule is *strategy-proof.*

In the n-person case, and as usual, the dictatorial conclusion does not hold unless additional requirements are imposed on rules. Indeed, suppose that $N \equiv \{1, 2, 3\}$ and consider the rule that assigns to agent 2 the entire circle if agent 1 prefers $[0, \frac{T}{2}]$ to its complement, and assigns to agent 3 the entire circle otherwise. This type of rules, which are obviously *strategy-proof*, are suggested by Satterthwaite and Sonnenschein (1981) on other domains (but they violate the requirement of “non-bossiness”, which says that an agent should not be able to affect what other agents receive without affecting what he receives). Characterizing the entire class of *strategy-proof* and *non-bossy* selections from the Pareto solution in the n-person case is an open question.

8 Concluding comment

The cake division problem discussed here is closely related to the problem of dividing an interval. A circle differs from an interval only in that the extreme points are topologically identified. Yet, we have discovered that this apparently minor difference has a significant impact on the conclusions. As is the case for a number of other models, circularity tends to significantly complicate matters. This general lesson is confirmed here, but we emphasize that it is mainly when no-envy is chosen as fairness notion. We had good news to report for egalitarian-equivalence.

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