The Welfare Theorems and Economies with Land and a Finite Number of Traders

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AND

ECONOMIES WITH LAND AND A FINITE NUMBER OF TRADERS

By

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ABSTRACT

The welfare theorems when land is modelled as measurable subsets of the plane are considered. Convexity of preferences is not a natural assumption in this context. An example of a Pareto Optimum not supported by linear prices is given. The second welfare theorem with superadditive prices is proved. A counterexample to the first welfare theorem when prices are superadditive is given.

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I. Introduction

The literature pertaining to general equilibrium models with land has, for the most part, treated land (or location) as just another commodity (or index of commodities) in an abstract general equilibrium model. As a result, assumptions normally used for any other commodity are also made for land. These assumptions include homogeneity, divisibility, and convexity of preferences. For example, the local public goods literature (see Bewley (1981) for references) treats land like any other commodity. Location theory (see Wheaton (1979) or Beckmann (1969)) treats land like any other Arrow-Debreu-McKenzie commodity and location like any other commodity index. These axioms cannot be argued convincingly as being applicable to land.

The canonical model of location theory has a continuum of consumers distributed over geographical space such that each consumer values and owns a density of land (see, for example, Beckmann (1977) or ten Raa (1984)). Thus, utility is a function of a density of land, while location theorists take no exception to modelling consumers as a continuum. Hildenbrand (1974) and related papers provide the axiomatic underpinnings for the use of continuum models as mathematically convenient approximations to large but finite economies.

Location theory does not use this type of approach. If a land parcel is to be represented by a subset of a Euclidean space (say $\mathbb{R}^2$), then the σ-finiteness of the space implies that there is only a countable number of parcels of positive area in each partition of the space. As a consequence, a continuum of consumers must be endowed with and trade parcels of land of zero area (on average). Furthermore, any sequence of large but finite economies close to such a continuum economy has the property that average land area holdings must become close to zero, so that economies approximated by a continuum economy are pathological. Thus, the equilibria and comparative statics results of a continuum model are not necessarily close to those of any reasonable finite model. The densities of land in the continuum model cannot be interpreted as areas (unless the commodity space is not σ-finite), so consumers must have preferences over parcels of zero area; see Berliant (1985a) for proofs of these statements. Parcels of positive area always yield infinite utility. An alternative interpretation is that the continuum of agents represents fractions of individual consumers rather than individuals themselves. This interpretation has severe limitations as well (see Berliant and ten Raa (1986b)), as the aggregate behavior of a continuum of identical individuals is not necessarily the same as the behavior of one large individual.

The existence of equilibrium and the classical welfare theorems have not been examined in great depth for the monocentric city model, the standard continuum model used in urban economics and regional science. As in Berliant and ten Raa (1986b), it is easy to construct examples where all of the classical conditions, such as continuity and convexity of preferences, hold but no equilibrium exists. Thus, the first order conditions of this model, which are often used, can be vacuous. Berliant, Papageorgiou, and Wang (1986) find examples where the first or second welfare theorems fail even under the classical conditions.

The problems mentioned above can be remedied by assuming that there is a finite number of consumers and by modelling land as measurable subsets of $\mathbb{R}^2$ rather than as points in $\mathbb{R}^2$. In previous work with this model, one assumption that has been made is that utility for land can be represented by the integral or aggregation of a given marginal utility density. Necessary and sufficient conditions on preferences for such a utility representation are given in Berliant (1982). Implicit in such a representation is that parcels of land are not complements, since the utility from the union of two parcels is equal to the sum of the utilities of the two parcels. Thus, the closeness of coherence (as measured by, say, connectedness) of parcels cannot matter. Although such an assumption is quite strong, it is useful for developing techniques to deal with land. Under the assumption that consumers have such a utility, demand was characterized (Berliant (1984)), the existence of an equilibrium with L^1 prices was demonstrated, and the welfare theorems were proved (Berliant (1985b)). Berliant and ten Raa (1985a) examine demand in the context of more general preferences. Dunz (1984) took a game-theoretic approach to the same type of economy. It was demonstrated that the core of such an economy could be empty if non-L^1 utilities are admissible. Moreover, conditions more general than the restriction to integral utilities that guarantee a non-empty core are found.

In Berliant and Dunz (1983), preferences are only assumed to be represented by continuous (with respect to symmetric differencing) set functions. The existence of an equilibrium is proved provided no trader is indifferent between any two traders' endowments. The second welfare theorem is shown to hold without this additional assumption. These results are valid even though the assumptions on utilities are not enough to exclude the conterexample in Dunz (1984) showing that the core of the economy might be empty. Hence, it is obvious that the first welfare theorem can be false; this is the case even if the core is nonempty.

The reason for these rather unusual results is that prices are only assumed to have the same properties as utilities, i.e. they are continuous set functions. In a certain sense this is very reasonable, since prices are dependent on the bids of the traders, which in turn depend on utility functions, so the price function can be expected to have properties similar to the utilities. In another sense, this assumption is unreasonable since traders can separate or combine parcels and make a profit even in equilibrium. Thus, there are arbitrage possibilities in equilibrium. This would not be true if prices were additive.

The purpose of this paper is to explore the relationship between the welfare theorems and properties of prices. An example of a core allocation that cannot be supported by additive prices is presented. It shows that the second welfare theorem is false if prices are required to be additive. It also motivates the main result of this paper: showing the second welfare theorem is true for superadditve prices if no trader is indifferent between any two traders' Pareto efficient allocations, utility is monotone and if utility satisfies a Lipschitz condition. Not surprisingly, the first welfare theorem can fail when prices are only superadditive; an example is given. This example also has an equilibrium with additive prices, so its allocation is Pareto Optimal.

The main mathematical contribution of this paper is a superadditive set function extension theorem. That is, if a set function is continuous and superadditive on a subspace, conditions are found under which this set function can be extended in a continuous and superadditive manner to the whole space. This technique could be of use in the field of utility theory and revealed preference theory to recapture superadditive utilities.

The work is organized as follows. Section II contains the assumptions and definitions used in the model, Section III contains a counterexample to the second welfare theorem requiring additive prices, Section IV contains the second welfare theorem with superadditive prices, Section V contains a counterexample to the first welfare theorem with superadditive prices, while Section VI concludes.

II. The Model

There are N traders (N integer and finite) indexed by i, j, and k. For the sake of simplicity, it is assumed that land is the only commodity in the economy. Land is represented by a measurable subset of $\mathbb{R}^2$ called L. The dimension of L is unimportant, but we wish to be specific for examples, so $L \subseteq \mathbb{R}^2$. Assume L has finite Lebesgue measure. The consumption set of each trader is the collection of all measurable subsets of L called β. This means that land can be subdivided and recombined by traders in virtually any manner. Furthermore, anything immobile, such as houses, can be embedded in the land; the preferences used below capture this idea.

Next, it is necessary to impose a topology on β. Since Urysohn's lemma is implicitly applied to subspaces of β on several occasions, β must be a perfectly normal space. The easiest way to impose this type of topology is to make β into a metric space.

Any metric on β will work since the proofs do not rely on a particular metric. Of course, the metric that is chosen determines the continuity restriction on preferences. From our point of view, one natural metric space structure to impose is the following:

For A, B $\in \beta$, the distance

$$d(A,B) = m(A \Delta B)$$

where Δ is the symmetric difference of sets and m is Lebesgue measure on $\mathbb{R}^2$. If sets in β are identified with their indicator functions, d is the same as the L^1 metric on indicators. We call the topology on β generated by d the d-topology. This metric is used explicitly below, although others can be substituted with a slight modification of the hypotheses of the theorem.

Next, it is assumed that trader i has a utility U_i defined over β. It is also assumed that U_i is continuous with respect to the d-topology. The continuity of utilities with respect to the measure of the symmetric difference of parcels is one possible assumption to make on preferences or utilities over parcels. Given that preferences are d-continuous, sufficient conditions on preferences to generate such a utility can be found in Berliant (1986). Since the values of utilities depend on the particular B $\in \beta$ under examination, they can also depend on immobile objects in B (provided that this dependence is d-continuous).

The formal structure of the model is now provided:

<u>Definitions</u>: An <u>allocation</u> is a partition of L into measurable subsets, $(B_1,B_2,\ldots,B_N)$, one for each trader. That is, $\bigcup_{i=1}^{N} B_i = L$, $B_i \cap B_j = \phi \;\forall\; i \neq j$, and $B_i \in \beta \;\forall\; i$, i=1,2,...,N.

An <u>economy</u> is a (2N+1)-tuple $\{\beta; E_1, E_2, \ldots, E_N; U_1, U_2, \ldots U_N\}$, where β is a σ-algebra (each trader's consumption set), $(E_1, E_2, \ldots, E_N)$ is an allocation with $m(E_i) > 0 \; \forall \; i$ ($E_i \in \beta$ is trader i's endowment), and $U_i\colon \beta \to \mathbb{R}$ is a d-continuous function (trader i's utility).

An allocation $(B_1, B_2, \ldots, B_N)$ is <u>Pareto Efficient</u> if and only if there is no allocation $(C_1, C_2, \ldots, C_N)$ such that

$$U_i(C_i) \geq U_i(B_i) \; \forall \; 1 \leq i \leq N$$

and

$$U_j(C_j) > U_j(B_j) \text{ for some } j.$$

An allocation $(B_1, \ldots, B_N)$ is in the <u>core</u> if there is no $N' \subseteq \{1,2,\ldots,N\}$ and $C_i \in \beta$ for $i \in N'$ such that

$$\bigcup_{i \in N'} C_i = \bigcup_{i \in N'} E_i \text{ and } U_i(C_i) > U_i(B_i) \forall_i$$

A <u>price system</u> P is a d-continuous map $P\colon \beta \to \mathbb{R}$ that assigns a price to each parcel. If we use the indicators of sets to embed β in L^∞, note that P is not necessarily a linear functional (or integral) on elements of β. A price system P is called <u>superadditive</u> if $\forall S, T \in \beta$ with $S \cap T = \phi$, $P(S \cup T) \geq P(S) + P(T)$.

An <u>equilibrium relative to a price system</u> P is an allocation $(B_1, B_2, \ldots, B_N)$ such that $\forall$ i, $U_i(C) > U_i(B_i)$ implies $P(C) > P(B_i)$ if $C \in \beta$.

An <u>equilibrium</u> relative to an economy is a price system and an allocation $(P; B_1, B_2, \ldots B_N)$ such that $U_i(C) > U_i(B_i)$ implies $P(C) > P(E_i)$ if $C \in \beta$, and $P(B_i) \leq P(E_i) \; \forall \; i, \; 1 \leq i \leq N$.

The definitions given above are all analogous to the standard definitions.

Notation: \ is set subtraction. Greek letters represent subsets of β while capital letters represent elements of β. If $B \in \beta$, B^c is the complement of B in L.

III. Second Welfare Theorem with Additive Prices

In this section we present a simple two person exchange economy with a core allocation that is not an equilibrium relative to any additive price function. The idea behind the example is very simple. The utility of the agents will depend on the measure of land in each part of a 2 element partition of the space of land. This means that there are essentially two homogeneous commodities. So the example will be equivalent to a standard 2 x 2 exchange economy. In such an economy, it is well-known that the second welfare need not hold when preferences are not convex. This is the case in our example. Note that when the commodity is land, convexity of preferences is not necessarily a natural assumption. For example, think of the partition as representing land on the East and West coast. An agent might be indifferent between 1 acre parcels on either coast, but prefer either of these parcels to having 1/2 acre on both coasts (see Schweizer, Varaiya, and Hartwick (1976)).

Let (A,B) be a measurable partition of the space of land with $m(A) = m(B) = 1$. Define the utiltiy functions by $U_1(S) = m(B \cap S) + \frac{1}{2}[m(A \cap S)+\frac{1}{2}]^2$ and $U_2(S) = [m(A \cap S)+\frac{1}{2}][m(B \cap S)+\frac{1}{2}]$.

Pick C_1 such that $m(C_1 \cap A) = m(C_1 \cap B) = \frac{1}{2}$ and let $C_2 = C_1^c$. It is easy to verify that the allocation (C_1, C_2) is in the core. To see this, look at the associated 2 x 2 exchange economy where (C_1, C_2) is represented by the vector $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$, i.e. $((m(A \cap C_1), m(B \cap C_1)), (m(A \cap C_2), m(B \cap C_2)))$. Notice first that neither one-person coalition will block this allocation. To see that it is Pareto efficient, compute $U_2(S)$ given $U_1(T) = 1$ and (T,S) is an allocation. The expression obtained is $U_2(S) = \frac{13}{16} + \frac{7}{8} m(A \cap S) - \frac{5}{4}[m(A \cap S)]^2 + \frac{1}{2} [m(A \cap S)]^3$. The first order condition for maximization of U_2 yields

$$\frac{7}{8} - \frac{5}{2} m(A \cap S) + \frac{3}{2} [m(A \cap S)]^2.$$

It is easily verified that the zeros of this equation are at $m(A \cap S) = \frac{1}{2}$, $m(A \cap S) = \frac{7}{6}$, and that the slope of $U_2(S)$ is positive for $m(A \cap S) < \frac{1}{2}$ and negative for $m(A \cap S) > \frac{1}{2}$, so $m(A \cap S) = \frac{1}{2}$ is the global maximum. Hence the allocation is Pareto efficient. In addition, agent 1 has nonconvex preferences in this economy and so it is clear that there do not exist prices of land in A and land in B that support this alloction. The following shows this formally.

Suppose P is additive and supports (C_1, C_2). We show this yields a contradiction. Consider the following trades agent 1 can make. Let $S \subseteq C_1$ and $T \subseteq C_1^c$ be such that $U_1(T \cup C_1 \setminus S) = \frac{1}{2} - m(S) + \frac{1}{2}(1+m(T))^2 > U_1(C_1) = 1$. So agent 1 prefers to give up such $S \subseteq A$ for $T \subseteq B$. Since P supports (C_1, C_2), $P(S) > P(T)$ for all S,T satisfying the above inequality. Now let $S \subseteq C_1^c$ and

$T \subseteq C_1$ be such that $U_1(S \cup C_1 \setminus T) = \frac{1}{2} + m(S) + \frac{1}{2}(1-m(T))^2 > U_1(C_1) = 1$. So agent 1 prefers to give up such $T \subseteq B$ for $S \subseteq B$ and so $P(T) > P(S)$ for such S,T satisfying the above inequality. Finding S,T that satisfy both inequalities yields the desired contradiction. This is true for $m(S) = m(T) = .1$.

The theorem in the next section shows that, although additive supporting prices might not exist, under relatively weak conditions superadditive supporting prices exist. This theorem covers this example and so there are superadditive prices supporting (C_1, C_2).

IV. Second Welfare Theorem with Superadditive Prices

<u>Theorem</u>: If $(B_1, \ldots, B_N)$ is a Pareto efficient allocation such that $\forall i$, $m(B_i) > 0$ and U_i satisfies:

$\forall i \; \forall j \; \forall k \neq j$, $U_i(B_j) \neq U_i(B_k)$; (1)

$\exists a,b$ such that $b > a > 0$ and $\forall S, S' \in \beta$

with $S \neq S'$ and $S \subseteq S'$, $a < \frac{|U_i(S)-U_i(S')|}{d(S,S')} < b$; (2)

if $S' \subseteq S$ with $m(S') < m(S)$ then $U_i(S') < U_i(S)$ (3)

then there exists a superadditive price system, P, such that $(B_1, \ldots, B_N)$ is an equilibrium relative to P.

Condition (1) says that no trader is indifferent between any elements of the Pareto efficient allocation. It is similar to the lack of indifference used by Roth and Postlewaite (1977). Condition (2) is a Lipschitz condition. It says the additional utility (per unit of land) received from having strictly more land is bounded from above and below. Note that since there are only finitely many traders, writing the Lipschitz bounds, a and b, as independent of i is not a restriction. Condition (3) implies that preferences are monotone. The proof below is constructive in that prices are found explicitly.

Proof: Without loss of generality we can assume that $\forall i$, $U_i(\phi) = 0$. We also introduce a dummy trader, 0, with endowment $B_0 = \phi$ and utility function $U_0(S) = m(S)$.

We begin by partitioning the set of traders, $\{0,1,\ldots,N\}$, into $M + 1$ sets (where M has yet to be determined). This partition is produced using the following inductive procedure.

Let $J \subseteq \{0,1,\ldots,N\}$ be the set of all agents who have not been assigned to an element of the partition up to this point in the induction. It is claimed that there exists $j \in J$ such that $U_j(B_j) \geq U_j(B_i)$ for all $i \in J$. If this were not true, then there would exist a cycle of indices, $\{i_1,\ldots,i_k\}$, $i_\alpha \in J \forall \alpha$, such that $U_{i_\alpha}(B_{i_{\alpha+1}}) > U_{i_\alpha}(B_{i_\alpha}) \forall \alpha$, where $i_{k+1} \equiv i_1$. But this implies that there is a reassignment of the B_i (assigning $B_{i_{\alpha+1}}$ to i_α and B_i to i for i not in the cycle) that yields an allocation Pareto superior to $(B_1,\ldots,B_N)$. A contradiction results since $(B_1,\ldots,B_N)$ is Pareto efficient. Therefore, $\exists j \in J$ such that $\forall i \in J$, $U_j(B_j) > U_j(B_i)$. Let $A_n \equiv \{k \in J \mid U_k(B_k) \geq U_k(B_i) \forall i \in J\} \neq \emptyset$, where n is the number of steps in the induction so far. For the next inductive step, we define the new set of remaining agents to be $J \backslash A_n$ and proceed as before. This procedure ends in at most $N + 1$ steps. Let M be the number of sets, $A_1, A_2, \ldots, A_{M+1}$ produced by this procedure. Note that by definition of the dummy trader's characteristics, $A_{M+1} = \{0\}$.

Now define $\beta_k \equiv \{B\epsilon\beta | U_i(B) \geq U_i(B_i)$ for some $i\epsilon A_k\}$ for $1 \leq k \leq M + 1$. For all $k \leq M$ and $i\epsilon A_k$, define $s_{ik} \equiv \inf \{m(B \cap B_j^c) | U_i(B) \geq U_i(B_i), j\epsilon A_h, j \neq i, h \geq k\}$ and $s \equiv \min_{1\leq k\leq M} \min_{i\epsilon A_k} s_{ik}$. The existence of the dummy trader, the definition of A_k, and (1) guarantee that $s_{ik} > 0$ for all i,k. Without this extra agent, s_{iM} could be zero if A_M had only one trader. These s_{ik} will be used to define equilibrium prices restricted to each β_k, $k \leq M$.

Let $m \equiv \min\{m(B_i) | 1 \leq i \leq M\}$. Pick $t > 2b(M+1)/am$ and $r > \max\{2b(M+1)/sam, 2b/am^2\}$. Now for $1 \leq k \leq M$, define $g_k : \beta \to \mathbb{R}_+$ by

$$g_k(B) \equiv rf_k(B) + \sum_{i\epsilon A_k} \frac{m(B \cap B_i)}{m(B_i)} + tm(B \cap [L \setminus \bigcup_{j\epsilon A_k} B_j])$$

where $f_k(B) \equiv \sum_{i\epsilon A_k} m(B \cap B_i)m(B \cap B_i^c)$.

We claim, $\forall k$,

g_k is continuous and superadditive, (4)

$g_k(B_i) = 1$ for $i\epsilon A_k$, and (5)

$g_k(B) > 1$ $\forall B\epsilon\beta_k$ such that $B \neq B_i$ for all $i\epsilon A_k$. (6)

Continuity of g_k is obvious. For $B,C\epsilon\beta$ such that $B \cap C = \phi$, $f_k(B\cup C) = f_k(B) + f_k(C) + \sum_{i\epsilon A_k} [m(B \cap B_i)m(C \cap B_i^c) + m(C \cap B_i)m(B \cap B_i^c)]$. Therefore, since the other terms of g_k are additive, each g_k is superadditive.

To see that $g_k(B_i) = 1$ for all $i\epsilon A_k$ notice that all terms of $g_k(B)$ are zero when $B = B_i$ except for $m(B_i \cap B_i)/m(B_i) = 1$. The terms in $f_k(B)$ disappear since $m(B_i \cap B_j) = 0$ $\forall i,j\epsilon A_k$, $i \neq j$. Next it is shown that $g_k(B) > 1$ $\forall B\epsilon\beta_k$ with $B \neq B_i$ for all $i\epsilon A_k$. Pick $B\epsilon\beta_k$ with $B \neq B_j$ for all $j\epsilon A_k$ and let $i\epsilon A_k$ be such that $U_i(B) \geq U_i(B_i)$. Such an i exists by the definitions of

A_k and β_k. By the continuity of U_i and (3), $U_i(B) \geq U_i(B_i)$ implies that $B \not\subset B_i$. If $B_i \subset B$ with $m(B_i) < m(B)$ then it is easy to see that $g_k(B) > 1$. This leaves one other possibility: $\delta \equiv m(B \backslash B_i) > 0$ and $\epsilon \equiv m(B_i \backslash B) > 0$.

Let $B' \equiv B_i \cap B$. Then by (2) and since $U_i(B') \leq U_i(B_i) \leq U_i(B)$, $a\epsilon = ad(B_i, B') < |U_i(B_i) - U_i(B')| \leq |U_i(B) - U_i(B')| < bd(B,B') = b\delta$. So $\delta > a\epsilon/b > 0$. To show $g_k(B) > 1$ we consider two cases: $\epsilon < m(B_i)/2$ and $\epsilon \geq m(B_i)/2$.

If $\epsilon < m(B_i)/2$ then

$$
\begin{aligned}
g_k(B) &\geq rf_k(B) + m(B \cap B_i)/m(B_i) \\
&\geq r(m(B_i)-\epsilon)\delta + (m(B_i)-\epsilon)/m(B_i) \\
&> r(m(B_i)/2)(a\epsilon/b) + 1 - (\epsilon/m(B_i)) \\
&> 1 \text{ since } r > 2b/a[m(B_i)]^2 \text{ by definition.}
\end{aligned}
$$

If $\epsilon \geq m(B_i)/2$ then $\delta > am(B_i)/2b$. Either $\exists j \epsilon A_k$, $j \neq i$ with $m(B \cap B_j) \geq \delta/(\#A_k+1)$ or $m(B \cap [L \backslash \bigcup_{h \epsilon A_k} B_h]) \geq \delta/(\#A_k+1)$ where $\#A_k$ is the cardinality of A_k. Suppose the former is true. Then, since $U_i(B) \geq U_i(B_i)$ implies $m(B \cap B_j^c) \geq s$, $g_k(B) \geq rf_k(B) \geq rm(B \cap B_j)m(B \cap B_j^c) \geq r\delta s/(\#A_k+1) \geq rsam(B_i)/[2b(\#A_k+1)] > 1$ by the definition of r. In the other case, $g_k(B) \geq tm(B \cap [L \backslash \bigcup_{h \epsilon A_k} B_h]) \geq t\delta/(\#A_k+1) \geq tam(B_i)/2b(\#A_k+1) > 1$ by the definition of t. So $g_k(B) > 1$ for all $B \epsilon \beta_k$ with $B \neq B_j$ for $j \epsilon A_k$. Hence, we have demonstrated (4), (5), and (6).

Next, we construct the supporting superadditive price function. Initially, let $P(B) \equiv g_1(B)$ for $B \epsilon \beta$. The properties of g_1 imply that P is supporting for all agents in A_1, i.e. $\forall i \epsilon A_1$, $\forall B \epsilon \beta$, $U_i(B) > U_i(B_i) \rightarrow P(B) > P(B)$. Also, $\forall B \epsilon \beta$, $P(B) \geq m(B) \cdot \min\left(\frac{1}{\bar{m} \cdot N}, \frac{t}{N}\right) = km(B)$ for $k > 0$, where $\bar{m} = \max_{1 \leq i \leq N} m(B_i)$. The construction of the supporting price function, P, proceeds by induction.

Given a superadditive price function, P, such that for some $k > 0$ $P(B) \geq km(B)$ $\forall B \in \beta$ and $\forall i \in \bigcup_{k<n} A_k$, $U_i(B) > U_i(B_i) \rightarrow P(B) > P(B_i)$, step n defines a new superadditive price function P' such that $\forall i \in \bigcup_{k \leq n} A_k$, $U_i(B) > U_i(B_i) \rightarrow P'(B) > P'(B_i)$. Also, for some $k' > 0$ $P'(B) \geq k'm(B)$ $\forall B \in \beta$. So P must be extended to $\beta_n \setminus \bigcup_{k<n} \beta_k$ in a way that preserves superadditivity and continuity. If $n = M+1$, let $P' = P$, and we are done.

Now $\forall$ $i \in A_k$, $k < n$, $\forall$ $j \in A_{k'}$, $k' \geq n$, $U_i(B_j) < U_i(B_i)$. Hence $\exists \epsilon > 0$ such that $U_i(B_i) - \epsilon > U_i(B_j)$ for all such i and j. Define $I_n \equiv \{B \in \beta_n | U_i(B_i) - \frac{\epsilon}{2} \leq U_i(B)$ for some $i \in A_k$, $k<n\}$. $\forall$ $j \in A_{k'}$, $k' \geq n$, $B_j \notin I_n$. $\forall$ $i \in A_k$, $k > n$, $B_i \in I_n$. Note that the case $\bigcup_{k<n} \beta_k = I_n$ is allowed; there is a gap in the consumption set when this happens. If $\mathcal{S} \subseteq \beta$, $B \in \beta$, define $d(B,\mathcal{S}) = \inf_{B' \in \mathcal{S}} d(B,B')$. If $I_n \neq \bigcup_{k<n} \beta_k$, then $\sup\{d(S, \bigcup_{k<n} \beta_k) | S \in I_n\} > 0$. Notice that $\forall$ $B \in \beta$, $g_n(B) \leq \frac{rN}{4}[m(B)]^2 + N \frac{m(B)}{m} + tm(B)$. Hence $v \equiv \inf\{\frac{P(B)}{g_n(B)} | B \in \beta,\ m(B) > 0\} > 0$

Define $\alpha: \beta \rightarrow \mathbb{R}$ by the following:

$$\alpha(B) \equiv \begin{cases} 0 & \text{for } B \in \bigcup_{k<n} \beta_k \\ d(B, \bigcup_{k<n} \beta_k)/\sup\{d(S, \bigcup_{k<n} \beta_k) | S \in I_n\} & \text{for } B \in I_n \\ 1 & \text{otherwise} \end{cases}.$$

Now define new prices

$P'(B) = (1-\alpha(B)) \frac{1}{v} P(B) + \alpha(B) g_n(B)$. Note that $P'(B) = \frac{1}{v} P(B)$ on $\bigcup_{k<n} \beta_k$ and that $\frac{1}{v} P(B) \geq g_n(B)$ $\forall$ $B \in \beta$.

This P' is clearly continuous; if $I_n = \phi$, then there is a gap between β_n and $\bigcup_{k<n} \beta_k$. $P'(B) > 0$ for all $B \in \beta$ with $m(B) > 0$. Also, by definition of g_n and α, it is clear that P' supports B_i $\forall i \in \bigcup_{k \le n} A_k$. Also, $\forall$ $B \in \beta$, $P'(B) \ge g_n(B) \ge \min(\frac{1}{mN}, \frac{t}{N})m(B)$. So all that remains to be checked is that P' is superadditive. Let $B, C \in \beta$ with $B \cap C = \phi$. Note that $S \subseteq S'$ implies $\alpha(S') \le \alpha(S)$. This fact along with the superadditivity of P and g_n imply the following sequence of inequalities:

$$
\begin{aligned}
P'(BUC) &\equiv [1-\alpha(BUC)]\tfrac{1}{v}P(BUC) + \alpha(BUC)g_n(BUC) \\
&\ge [1-\alpha(BUC)]\tfrac{1}{v}[P(B)+P(C)] + \alpha(BUC)[g_n(B)+g_n(C)] \\
&= [1-\alpha(BUC)]\tfrac{1}{v}P(B)+\alpha(BUC)g_n(B) + [1-\alpha(BUC)]\tfrac{1}{v}P(C)+\alpha(BUC)g_n(C) \\
&\ge [1-\alpha(B)]\tfrac{1}{v}P(B)+\alpha(B)g_n(B) + [1-\alpha(C)]\tfrac{1}{v}P(C)+\alpha(C)g_n(C) \\
&\equiv P'(B) + P'(C)
\end{aligned}
$$

The second to last step follows from $\frac{1}{v}P(B) \ge g_n(B)$ for $B \in \beta$. So P' is superadditive. Now, let $P(B) = P'(B)$ and proceed to the next inductive step. After the last step, we have a continuous, superadditive P such that $\forall_i, U_i(B) > U_i(B_i) \rightarrow P(B) > P(B_i)$ and therefore $(B_i, \ldots, B_N)$ is an equilibrium relative to P. Q.E.D.

V. The First Welfare Theorem

Berliant and Dunz (1983) show that the first welfare theorem is false if no restrictions other than continuity are imposed on the equilibrium price function. Here we show that the first welfare theorem fails even when prices are superadditive.

Let L be the square with side length 1 in $\mathbb{R}^2$ centered at (0,0). β is the set of measurable subsets of L. Let K_1 be the set of points with negative first coordinate and let K_2 be the set of points with non-negative first coordinate. There are two traders, one who likes K_1 and one who likes K_2. Their utilities are given by $U_1(B) = m(B \cap K_2) + 2m(B \cap K_1), U_2(B) = m(B \cap K_1) + 2m(B \cap K_2)$ for $B \in \beta$. Let the endowment of trader 1 be $E_1 = \{(x_1,x_2) \in L | x_2 \geq \frac{1}{4}\}$ and let the endowment of trader 2 be $E_2 = L \backslash E_1$. Since the utilities can be expressed as integrals, Berliant (1985b) applies and there is a Pareto Efficient equilibrium with a price function that can be expressed as an integral.

The following superadditive price function, P, results in (P,E_1,E_2) as an equilibrium:

$$P(B) = m(B \cap E_2) + 16m(B \cap E_2)m(B \cap E_1) + \frac{1}{2} m(B \cap E_1).$$

To see that this is an equilibrium, consider each trader's utility-increasing trades. Trader 1 must get more than $\frac{1}{2}$ unit of land in E_2 for every unit of land in E_1 released in order to obtain more utility. However, P enables him to buy only at most $\frac{1}{2}$ unit of land in E_2 for every unit in E_1. Therefore, everything preferred by trader 1 costs more than his endowment. In order to obtain more utility, trader 2 must also trade land in E_2 for land in E_1 at a ratio less than 2:1. Consider a parcel B that trader 2 gets by giving up a subset of $E_2 \cap K_1$ of measure k and receiving a subset of $E_1 \cap K_2$ of measure greater than $\frac{1}{2}$ k. Note that $k < \frac{1}{4}$ since $m(E_1 \cap K_2) = \frac{1}{8}$. So

$$\begin{aligned} P(B) &> (\tfrac{3}{4} - k) + 16(\tfrac{3}{4} - k)(\tfrac{1}{2} k) + \tfrac{1}{2}(\tfrac{1}{2} k) \\ &= \tfrac{3}{4} - \tfrac{3}{4} k + 6k - 8k^2 \\ &= \tfrac{3}{4} + k(5 \tfrac{1}{4} - 8k) \\ &> \tfrac{3}{4} = P(E_2) \text{ since } k > 0 \text{ (otherwise } U_2(B) \leq U_2(E_2)) \text{ and } 5 \tfrac{1}{4} - 8k > 0. \end{aligned}$$

Therefore, (P,E_1,E_2) is an equilibrium. However, (E_1,E_2) is not Pareto efficient since it is Pareto dominated by the allocation $B_1 \equiv \{(x,y) \epsilon L | x < 0, y < 0\}$ and $B_2 = L \backslash B_1$.

VI. Conclusions

The model presented above is unusual in that the commodity space has no linear structure. This is natural in the context of land, and might be natural in other contexts as well when one can simply put all units of all commodities out on the lawn and allow traders to purchase subsets. In fact, the restriction of L to $\mathbb{R}^2$ is unnecessary; all that is required is a metric space structure on a σ-algebra.

The failure of the first welfare theorem to hold in such models is not surprising. The reason is that equilibrium prices are not required to be additive. This means that certain kinds of arbitrage trades can increase a trader's utility. So it seems that no arbitrage implies additive prices (see Berliant and ten Raa (1986a)). However, the example in Section III shows that additive prices cannot support all Pareto efficient allocations. The example also shows that an equilibrium with additive prices need not exist. If the endowments are the efficient allocation of this example, then the only possible equilibrium is the no trade equilibrium. But additive prices cannot support this allocation. So in such a situation equilibrium requires nonadditive prices. The existence of arbitrage possibilities at such an equilibrium casts doubt upon whether it should really be called an "equilibrium".

Exactly what kind of final allocations and prices are likely to result in the economies with land described in this paper is far from resolved. There seem to be at least two natural ways to proceed. One is to look for further assumptions on preferences (weaker than the additivity used in Berliant (1982, 1985b)) sufficient to guarantee the existence of equilibria with additive prices. Alternatively, game-theoretic or bargaining solution concepts could be considered. This is natural since each parcel of land can be essentially unique and, therefore, price-taking behavior might not be a reasonable assumption.

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