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Abstract

This essay surveys the literature on the axiomatic model of bargaining formulated by Nash ("The Bargaining Problem," *Econometrica 28*, 1950, 155-162).

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Key-words; Nash's bargaining model. Nash solution. Kalai-Smorodinsky solution. Egalitarian solution.

Contents

1 Introduction

Almost sixty years ago, Nash (1950) published his seminal paper on what is now known as the "axiomatic theory of bargaining". The "bargaining problem" formalizes the following situation. Two agents have access to any of the "alternatives" in some "feasible set". Their preferences over these alternatives differ. If they reach a compromise on a particular alternative, that is what they get. Otherwise, they end up at a pre-specified alternative in the feasible set, the "disagreement point". The goal is to predict how they would settle their differences, or for another interpretation of the model, how an impartial arbitrator would identify a fair compromise to recommend to them.

To illustrate, consider the negotiations between the management and the workers of a firm. Here, the feasible alternatives consist of specifications of salaries, benefit packages, working conditions, and so on. Disagreement results in a strike, an outcome that is costly to both sides.

Nash specified a class of conflict situations of this type, and he defined a "bargaining solution" as a function that produces, for each problem in the class, an alternative in that problem. He formulated a list of properties, or "axioms", that he thought a solution should satisfy, and he established the existence and the uniqueness of a solution satisfying all of the axioms; this solution is now called the "Nash solution". Nash limited his attention to the two-person case but his solution can easily be extended to the n-person case; so can his axioms and his characterization. Thus, the expression "Nash solution" is used independently of how many agents are involved.

Nash (1953) also suggested that each bargaining problem could be analyzed as a non-cooperative game. He proposed a way of associating with the problem a set of strategies for each agent, and specified a function assigning a payoff vector to each profile of strategies; then, he searched for profiles of strategies such that each agent's strategy is a best response to the strategy chosen by the other, such a profile being a "Nash (non-cooperative) equilibrium" of the game. He then asked whether the outcome attained at these profiles corresponded to the outcomes obtained by applying the solution he had derived axiomatically. He identified certain conditions under which the answer is yes.

Nash's model has been one of the most successful paradigms of game theory. His paper is the founding stone of a literature that now comprises several hundred theoretical papers.[1] The Nash solution is presented in all game theory textbooks (for example, see Osborne and Rubinstein, 1990), and an important section of the leading microeconomics manual is devoted to it (Mas-Colell, Whinston, and Green, 1994). Together with the Shapley value (Shapley, 1953) and the core (Gillies, 1959), it constitutes the obligatory background on cooperative games in most economics graduate programs. (Yaari, 1981, relates Nash's theory to other normative theories of resource allocation.) It has also been applied in countless empirical studies.

Nevertheless, over the years, it has faced many challenges, and one cannot say that Nash "solved" the bargaining problem. His axiomatic treatment suffers from a number of limitations, and his strategic model delivers the outcome predicted by his axiomatic formulation only under certain assumptions that are not all totally compelling.

The present volume collects papers in which the various axiomatic approaches that have been followed in addressing the issues raised by Nash's formulation have been initiated and developed. We start with the "historical" papers and end with papers illustrating the latest trends in the literature.

For each particular perspective, it often occurred that several natural candidate papers could have been selected. Our main objective in making the difficult but necessary choices was that each of the main ideas be represented. We hope that, with the help of this introduction, each reader will be able to identify supplementary readings in the area of his or her interests.

We begin with Nash's own paper. This is the only paper that we discuss in any detail. We then present the criticisms to which it has been subjected, using them as a platform from which to jump in discussing subsequent work.

Nash viewed the cooperative and the strategic models as complementary, but here, we will mainly concern ourselves with the cooperative model. The strategic model has also spawn a considerable literature and a separate volume would be needed to do justice to it. Section 11 should help bridging the literatures however.

[1]For surveys, see Roth (1979), Kalai (1985), Thomson and Lensberg (1989), Peters (1992), Gaertner and Klemisch-Ahlert (1992), Klemisch-Ahlert (1996), and Thomson (1985a, 1994, 2008).

2 The formal model and Nash's axiomatic derivation of the Nash solution

There is a fixed set N of **agents**. Each agent is equipped with preferences defined over some underlying set of physical outcomes, and lotteries over these outcomes. Preferences satisfy the postulates of von-Neumann and Morgenstern (1944), and thus can be represented by functions satisfying a certain expectation formula—they are "von Neumann-Morgernstern utility functions". The agents have access to any utility vector obtained as the image of one of the physical outcomes, or lotteries over these outcomes. However, no information about this underlying set is retained in specifying a problem (an assumption that has been the most controversial in recent literature; Subsection 10.3).

A (bargaining) **problem** consists of a pair (S, d) where S, the **feasible set**, is a subset of $\mathbb{R}^N$ of **alternatives**, and $d \in \mathbb{R}^N$, the **disagreement point**, is a point of S. The set S is compact and convex, and there is at least one point of S strictly dominating d. To simplify, we consider the class of **comprehensive** problems: for such a problem, if a point x is in S, then so is any point y with $d \leq y \leq x$. A **strictly comprehensive** problem S is comprehensive and in addition, if a point x is in S and a point y satisfies $d \leq y \leq x$, then there is a point z in S such that $z > y$ (in words, the part of the northeast boundary of S that dominates d does not contain a segment parallel to a coordinate axis). A **fully comprehensive** problem S is unbounded below and if a point x is feasible, so is any point y such that $y \leqq x$. A subclass of problems that is particularly convenient in order to illustrate the central definitions, as it allows circumventing certain technical issues, is obtained by setting $d = 0$ and requiring S to be a compact, convex, and comprehensive subset of $\mathbb{R}^N_+$ containing at least one positive vector. A (bargaining) **solution** defined on a class of problems is a function that associates with each problem (S, d) in the class a unique point of S, the **solution outcome of (S, d)**.

The methodology we follow in searching for the "most desirable" solutions is axiomatic.[2] An **axiom** is the mathematical expression of our intuition of how a solution should behave in certain situations. Axioms come in two categories. A **punctual axiom** applies to each problem separately. A **relational axiom** relates the choices made by a solution as the data of the

[2]A general presentation of the axiomatic method and of its recent applications to game theory and resource allocation is Thomson (2001).

problem change in a certain way. An **axiomatic characterization** is a theorem identifying a particular solution (or a family of solutions) as the only one (or the only solutions) satisfying a particular list of axioms. Such a result is often proved by first handling problems with a simple structure by means of the punctual axioms, then settling the case of arbitrary problems by relating them to simple ones by means of the relational axioms.

Nash (1950) considered the two-agent case and required solutions to satisfy the following axioms:[3]

Pareto-optimality: there should be no alternative at which both agents' payoffs are at least as large as they are at the solution outcome, and at least one agent's payoff is larger.

Symmetry: if a problem is symmetric with respect to the 45° line, its solution outcome should be on that line.

Scale invariance: von Neumann-Morgenstern utilities being unique only up to positive affine transformations, the solution outcome should be invariant under such transformations.

Contraction independence: if, keeping the disagreement point constant, the feasible set contracts but the alternative chosen as solution outcome remains feasible, then it should remain the solution outcome.[4]

Nash proved that only one solution satisfies these axioms. It is the solution that selects, for each problem (S, d) the unique point of S at which the product of the agents' utility gains from d is the largest among all points of S dominating d. The solution is the **Nash solution**. The result easily generalizes to arbitrarily many agents.

Theorem 1 (Nash, 1950) *The Nash solution is the only solution satisfying Pareto-optimality, symmetry, scale invariance, and contraction independence.*

Following the scheme outlined above, the proof consists in solving symmetric problems by invoking *Pareto-optimality* and *symmetry*; then, appealing to *scale invariance* to transform an arbitrary problem into one whose disagreement point and Nash point both have equal coordinates, and finally,

[3]This presentation of the problem slightly differs from Nash's original formulation.

[4]The axiom appears in Nash's paper under the name of "independence of irrelevant alternatives", but we prefer a more neutral expression, one that does not prejudge of the irrelevance of the alternatives that are eliminated.

after "embedding" this normalized problem into a symmetric one, invoking *contraction independence*.

3 Did Nash solve the bargaining problem?

Nash's theorem provided an answer to the undeterminatedness of the terms of bargaining that had stumped previous writers. But was it *the* answer? An axiomatic characterization is as good as the axiom system on which it is based, and a number of criticisms of Nash's were soon raised: the axiomatic literature on the bargaining problem was launched.

From a normative viewpoint, it is difficult to see why *Pareto-optimality* should not be required, but this property is certainly not always observed in actual bargaining and it does not seem to be a necessary part of a descriptive theory. Nevertheless, the axiom plays a limited role in precipitating Nash's conclusion: if it is dropped, only one additional solution becomes acceptable. It is the trivial solution that always selects the disagreement point (Roth, 1977a, 1980. See Mariotti, 1994, 1996, 1999, 2000a, 2000b, for alternatives to *contraction independence* that also lead to the Nash solution).

Symmetry seems hard to quibble with: if nothing distinguishes the agents, on what grounds could one choose a point that is not symmetric? Of course, agents who enter symmetrically in a problem may not be "the same" in certain respects that are not explicitly modeled. From a descriptive viewpoint, agents may differ in attributes that may affect their ability to extract concessions from their opponents. From a normative viewpoint, an arbitrator may have his own reasons to want to favor certain agents over the others; instead of being individuals, agents may represent other kinds of entities, such as countries or families, that differ in their size, needs, rights, and so on. One can of course argue that if these factors are relevant, they should be incorporated into the model, but they may not be easily formalized or quantified. (For a discussion, see Schelling, 1959.) A merit of Nash's model is that they can instead be accommodated by simply dropping *symmetry* and investigating which additional solutions become available. One would expect any possible bias in favor of particular agents and against others to operate consistently across all problems, and this is indeed what the theory delivers, namely a parametric family of **weighted Nash solutions**, defined by maximizing the product of utility gains raised to powers that differ from agent to agent (Harsanyi and Selten, 1972). By choosing weights appropriately, the arbitrator can gear his recommendation in favor of a particular agent and against others, and to whatever extent he desires. Extreme violations of *symmetry* are accommodated by other solutions with "dictatorial" features (Peters, 1987b). A characterization when convexity of feasible sets is dropped is due to Peters and Vermeulen (2007).

Scale invariance appears compelling when the theory is meant to be descriptive but it precludes basing compromises on interpersonal comparisons of utility. In daily life, such comparisons are often made in reaching compromises. An arbitrator may similarly feel that these comparisons are relevant in making a recommendation.

Contraction independence has been the object of the sharpest criticisms. Nash himself expressed some misgivings about it and Luce and Raiffa (1957)'s objections are well-known. In evaluating a bargaining situation, it is unavoidable and probably desirable that it be simplified and summarized, that it be reduced to its essential features. The issue is how much and what information should be discarded in the process, and one can make a convincing case that *contraction independence* ignores too much. Indeed, it covers contractions that affect the shape of the feasible in ways that seem relevant, for instance, the elimination of only alternatives at which a particular agent's payoff is higher than at the initial solution outcome and the other agent's payoff lower than at the initial solution outcome; *contraction independence* prevents solutions from responding to such eliminations.

4 Beyond Nash

In spite of the criticisms levelled against Nash's axiom system, his solution has met with what can only be described an extraordinary success in economics, although it is probably the case that the profession at large did not adopt it for its axiomatic foundations, but rather for the ease with which it can be calculated: in order to determine whether an undominated alternative x is the Nash outcome of some problem (S, d), it suffices to calculate the slope of the boundary of S at x and check that it is the negative of the slope of the segment connecting x to d. Based on local information, the Nash solution is tailor-made for economists, steeped in marginal analysis.

Yet, recommending a payoff vector on the basis of marginal information seems odd. The goal is to balance the interests of different parties, comparing what each agent gets to what the others get at each particular alternative, and to what he and they could have gotten at other alternatives. As noted earlier, by altering the configuration of the alternatives available, one generally affects the attractiveness of a proposed compromise.

The main counterargument to this criticism was made by Nash himself: in practice, information is usually lacking about which alternatives are truly available, and a compromise under evaluation only competes against others that are not too different from itself (Nash wrote: there should be "no action at a distance"). Modeling this lack of information explicitly is what an investigator should perhaps do, but there are also advantages to keeping the model simple; *contraction independence* is a formal way to express the idea.

A great variety of perspectives have been taken over the years and a variety of axioms have been formulated reflecting these perspectives. To describe a problem, three entities are given, the set of alternatives, the disagreement point, and of course the population of agents itself. Accordingly, we organize our presentation by considering in turn how solutions take into account the feasible set, the disagreement point, and the population of agents, and how they respond to variations in this data. In later sections, we review extensions of the model, and discuss the importance of working in utility space and not retaining information about the structure of the physical alternatives from which payoff vectors are generated.

Some of these developments have led to the Nash solution, some not. That other solutions would have come out of this work is of course expected, but what may be surprising is the fact that, in spite of the large number of reasonable solutions that one can easily define, only three solutions, and variants, have consistently emerged: in addition to the Nash solution, they are the egalitarian and Kalai-Smorodinsky solutions. Weighted versions of these solutions, designed to accommodate desired biases in favor of particular agents (as discussed above), and for the latter two, lexicographic extensions, defined so as to recover *Pareto-optimality* when these solutions only deliver a weak form of the property (indeed the egalitarian and Kalai-Smorodinsky solutions do not satisfy *Pareto-optimality* as generally as the Nash solution does) should be added to the list (see below). The dominance of these three solutions and these variants is a central conclusion to be drawn from the literature.

The Nash solution has come out somewhat more often than the other two, and the claim can perhaps be made that it is indeed special; the mere accumulation of results in which it appears in the theory would be evidence in its favor. But the argument is a little dangerous. Earlier, we talked about the theorist's need to simplify and summarize in order to analyze, and in axiomatic analysis, simplification often takes the form of independence and invariance axioms. The Nash solution satisfying many invariance conditions, it is not much of a surprise that it should have come out often. On the other hand, the monotonicity axioms that have generally led to the Kalai-Smorodinsky and egalitarian solutions are readily understood and endorsed by the man on the street.

5 Other solutions

Before presenting new axioms, it is useful to attempt to solve bargaining problems directly, to use intuition to define compromises. Bargaining problems are relatively simple geometric objects and a wide range of solutions based on elementary geometric operations quickly come to mind. After introducing the issue in a classroom and taking care not to use language that might suggest particular resolutions—here, we do not pretend to have proceeded in a manner that experimental economists would fully endorse—we have often invited students to make proposals. Even with little or no background in economics or game theory, they had no trouble coming up with their own schemes, and several of the solutions defined below frequently showed up. We will let the reader decide whether the fact that the Nash solution itself has never been in their lists should be a surprise. An added reason for the landmark status enjoyed by Nash's paper (theory having revealing a well-behaved solution whose definition could not be easily guessed)? Or an additional challenge to this status (the meaning of a product of utility gains not being easily interpretable)? In the definitions below, (S, d) is an arbitrary n-agent problem.

The **egalitarian solution** (Kalai, 1977) selects the maximal point of S at which utility gains from d are equal. More generally, by making utility gains proportional to a fixed vector of weights, we obtain the **weighted egalitarian solution relative to these weights** (Kalai, 1977). Further, given a continuous and unbounded monotone path G in $\mathbb{R}^N_+$ emanating from the origin, the **monotone path solution relative to G** selects the maximal point of S on the path translated to d, namely $G + \{d\}$ (Thomson and Myerson, 1980). None of the egalitarian solution and of these generalizations is *Pareto-optimal*, but given the comprehensiveness assumption under which we are operating, they all satisfy the slightly weaker property of **weak Pareto-optimality**, which says that there should be no alternative in S at which all agents' payoffs are greater than they are at the solution outcome.

To recover full *Pareto-optimality*, we can perform a lexicographic operation: find the maximal point of S of equal utility gains from d, x^1. Identify the agents whose payoffs can be increased from x^1 without decreasing the payoff of any other agent, and then the maximal point of S at which these agents experience equal gains from x^1; repeat. At least one agent drops out at each step. Thus the process ends in finitely many steps. Obviously, the endpoint is Pareto optimal. This endpoint is the **lexicographic egalitarian solution** outcome. (If a problem is not comprehensive, work instead with its **comprehensive hull**, that is, the smallest comprehensive problem that contains it. This technique applies to several other solutions discussed below) This replacement does not affect the set of Pareto-optimal points. Lexicographic versions of the monotone path solutions can be defined in a similar way.

The **Kalai-Smorodinsky solution** (Kalai and Smorodinsky, 1975) selects the maximal point of S that is proportional to the profile of maximal payoffs that agents can separately reach among the points of S that dominate d. This list of maximal payoffs is the **ideal point** of (S, d). The Kalai-Smorodinsky solution is *Pareto-optimal* for two agents, but only *weakly Pareto-optimal* for more agents. It can be understood as a normalized version of the egalitarian solution, that is, a *scale invariant* version. The normalization consists in placing the ideal point on the line of equal gains from the disagreement point. Weighted and lexicographic versions of the solution can be defined in the way in which we defined such versions of the egalitarian solution.

The two-agent **equal loss solution** (Chun, 1988) selects the maximal point of S at which all agents' utility losses from the ideal point are equal. It involves a shift of perspectives, from the gains that agents achieve from d to the losses from the ideal point they experience. To extend the definition to n-agent problems, care must be exercised, as the line of equal losses may not meet the feasible set, or if it does, this meeting point may not dominate the disagreement point. To deal with these difficulties, lexicographic operations can be performed that preserve the spirit of the two-agent definition (Chun and Peters, 1991; Herrero and Marco, 1993). The equal loss solution belongs to a one-parameter family, the **Yu family** (Yu, 1973; Freimer and Yu, 1976), defined by minimizing the p-distance from the ideal point to the feasible set.

The member of the Yu family obtained for $p = 2$ is advocated by Salukvadze (1971a,b). Weighted versions of the solutions can also be defined.

The ***i*-th dictatorial solution** selects the payoff vector at which agent i's payoff is maximized subject to the other agents receiving their disagreement payoffs. Lexicographic versions of these solutions can be defined to recover *Pareto-optimality* as follows. Given a strict order on the set of agents, the **lexicographic dictatorial solution relative to that order** identifies the payoff vectors at which the payoff of the agent who is first in that order is maximized among the payoff vectors dominating d; among those, it identifies the payoff vectors at which the payoff of the agent who is second in the order is maximized; and so on, until only one payoff vector is left.

The **discrete Raiffa solution** (Raiffa, 1953) selects the limit of the sequence $\{x^t\}$ defined as follows: x^1 is the average of the dictatorial solution outcomes; x^2 is the average of the dictatorial solution outcomes obtained when x^1 is used as disagreement point, and so on. The **continuous Raiffa solution** selects the limit of the point $x(t)$ such that $x(0) = d$ and whose motion at time t is in the direction of the average of the dictatorial solution outcomes.

The two-agent **Perles-Maschler solution** (Perles and Maschler, 1981) selects the limit of the point $x(t)$ such that $x(0) = d$, but whose direction of motion at time t is based on comparing the relative marginal gains that agents achieve along the boundary of S at the dictatorial solution outcomes obtained when $x(t)$ is used as disagreement point. Generalizing the process to arbitrarily many agents is not straightforward however (Kolhberg, Maschler, and Perles, 1983; Calvo; Gutiérrez, 1994a; Rosenmüller, 2004). The Raiffa and Perles-Maschler solutions exemplify a family of solutions defined as limits of "concession" processes, starting from the agents' dictatorial outcomes. Another study of how solutions can be defined by specifying a direction of movement of $x(t)$ is Calvo and Peters (2000).

Each **utilitarian solution** (based on 19-th Century ideas) selects an alternative at which the sum of payoffs is maximal among all alternatives. In the present context, utilitarian objectives raise several difficulties. First, the maximizer may not be unique, which is why we have to speak of utilitarian solutions in the plural, having defined solutions as *single-valued* mappings. Unfortunately, selecting from the correspondence of maximizers cannot be done in a very satisfactory way. In particular, **continuity**, the requirement that small changes in problems should not lead to large changes in solution outcomes, has to fail. Second, the maximizer(s) is (are) independent of d

and it (they) may not even Pareto dominate d. This seems undesirable given the interpretation we gave to d. This second difficulty is easily dealt with however: simply maximize the sum of payoffs over the part of S that dominates d. The normalized version obtained by placing the ideal point on the line of equal gains from the disagreement point prior to maximizing the sum of payoffs is the **normalized utilitarian solution** (Cao, 1982). It uses the same information as the Kalai-Smorodinsky solution does in providing a normalization of the egalitarian solution.

A family of solutions obtained by maximizing a convex function that is linear on each of the rank-ordered subsets of utility space (the subsets on which payoffs are ordered in the same way) are introduced by Blackorby, Bossert, and Donaldson (1994) under the name of **generalized Gini solutions** because of their relation to the Gini index used in the theory developed for the measurement of income inequality. Further extensions are due to Ok and Zhou (2000)—they call them **Choquet solutions**—the latest step in this process of generalization being taken by Hinojosa, Mármol, and Zarzuelo (2008).

The two-agent **equal area solution** (Dekel, 1982; Ritz, 1985; Anbarci, 1993; 2006; Anbarci and Bigelow, 1993; In, 2008) selects the Pareto-optimal point x at which the area of S consisting of all points at which agent 1's payoff is greater than x_1 is equal to the area similarly defined for agent 2. This solution exemplifies a family obtained by calculating at each feasible point and for each agent, the "sacrifice" requested of him at that point, and selecting the point at which sacrifices are equal. Sacrifices can be measured in other ways. The two-agent **equal length solution** selects the Pareto-optimal point x at which the length of the curvi-linear segment in the Pareto-optimal boundary at which agent 1's payoff is greater than x_1 is equal to the similarly defined quantity for agent 2. For more than two agents, these definitions can be generalized by comparing volumes for the first solution, and comparing surface areas on the Pareto boundary for the second solution, but these definitions are not as compelling because an alternative at which an agent's payoff may be greater than his payoff at the solution outcome is not necessarily one at which all other agents' payoffs are at most as large as at the solution outcome.

6 Punctual axioms

We now turn to axioms. The transparent requirement of **individual rationality**, which we have implicitly invoked on several occasions already, says that at the solution outcome, all agents' payoffs should be at least as large as at d. **Strong individual rationality** says that they should be larger than at d. Even the strong form is met by most solutions, but surprisingly, if in Nash's theorem, *Pareto-optimality* is replaced by that axiom, the Nash solution remains the only acceptable one (Roth, 1977a).

Midpoint domination says that for each problem, each agent's payoff should be at least as large as the average of its dictatorial outcomes. This average can be interpreted as an equal-probability lottery over these outcomes. This axiom is quite weak but when complemented with *contraction independence*, only the Nash solution is acceptable (Moulin, 1983). A stronger (in fact, significantly stronger) version of the axiom is obtained by using the equal-probability lottery over the lexicographic dictatorial solution outcomes. The two-agent Perles-Maschler solution is the only one of the main solutions to satisfy this property, although others can certainly be defined that do. For two agents, Salonen (1985) defines such a solution, which can be seen as a version of the Kalai-Smorodinsky solution.

7 Relational axioms pertaining to the feasible set

• Basic axioms. A strengthening of *symmetry* is **anonymity**: the solution outcome of a problem should only depend on its geometry, not on the way in which the axes are named after agents. Most problems are not symmetric, so *symmetry* rarely applies, but *anonymity* always does because any problem can be subjected to the renaming operation described in the hypotheses of this axiom.

Independence of non-individually rational alternatives says that the solution outcome should only depend on the part of the feasible set that dominates the disagreement point.

Continuity says that small changes in problems should not lead to large changes in solution outcomes. There are several ways to define small changes in problems however, and to each of which corresponds a continuity notion (Peters, 1986; Livne, 1987; Salonen, 1996). Accommodating degenerate problems creates complications (Jansen and Tijs, 1983). Yet another notion can be defined in the context of a model in which population size may vary (Section 9; Lensberg, 1985a; Thomson, 1985b).

• Independence. A number of variants of *contraction independence* have been formulated. For example, the hypothesis of equal disagreement points in that axiom has been replaced by the assumption of equal "points of minimal expectations"—for a given problem, this is the point whose i-th coordinate is the utility achieved by agent i at the Pareto-optimal point at which agent j's payoff is maximized. A variant of the Nash solution obtained by maximizing the product of utility gains from the point of minimal expectations is defined and characterized along the lines of Nash's theorem by Roth (1977b). A formulation encompassing both Nash's and Roth's based on general "reference points" is proposed by Thomson (1981a). Another example of a reference point is the center of gravity of the individually rational region. This point also appear in a definition proposed by Radzik (1998). See also Conley, McLean, and Wilkie (1997) and Lahiri (1995). Other variants are discussed in Thomson and Myerson (1980) and Thomson (1981a). A recent contribution is Anbarci and Sun (2009).

Instead of contractions of feasible sets, one can also imagine expansions. Several formulations are possible, which we describe for solutions that are at least *weakly Pareto-optimal*, as they are most meaningful then. **Expansion independence** says that if a feasible set expands but the initial solution outcome remains weakly Pareto-optimal, it should remain the solution outcome. This is a strong requirement, but if the boundary of a problem is smooth at its solution outcome, it can be advocated by invoking the "localness" argument made by Nash to justify *contraction independence*. If the boundary has a kink at its solution outcome x, a problem containing x and for which x is weakly Pareto-optimal need not "look the same" as the initial problem around x. For two problems to look the same at a common boundary point x, the cones of their lines of support at x should be the same. Depending upon the complementary axioms, this formulation leads to the Nash solution (Thomson, 1981b), or to the utilitarian solution (Thomson, 1981c). The same logic suggests not imposing an inclusion relation between the two problems that are evaluated. A formal axiom of **localness** is formulated by Lensberg (1987) and also considered by Serrano and Shimomura (1998). One can also limit the expansion, for each problem, to problems whose feasible set is supported at the initial solution outcome, by a hyperplane that may depend on the initial problem.

An interesting question concerns the identification of properties of a solution that guarantee the existence of a binary relation defined over payoff space such that for each problem, the solution outcome is given as the sum of d and the maximizer of this relation over $S - \{d\}$. The relevance of *contraction independence* and of the axioms of revealed preference in answering this question is uncovered by Peters and Wakker (1987), Bossert (1994b), and Sánchez (2000). A related question is addressed by Bossert (1996). Rationalizability by tournaments (binary relations that may not be transitive) is examined by Lombardi and Mariotti (2009).

• Monotonicity. **Individual monotonicity** says that an expansion of the feasible set that is favorable to an agent should not hurt him. The idea is most easily described in the two-agent case. Suppose that the expansion leaves unaffected the maximal payoff that one of the agents, say agent 2, can reach when the other (agent 1) is assigned his disagreement payoff. Then, for each of agent 2's payoffs between his disagreement payoff and his maximal payoff as just calculated, the maximal payoff achievable by agent 1 is at least as large as it was initially. The requirement is that agent 1 should not be hurt. For two agents, the Kalai-Smorodinsky solution is the only one to be *Pareto-optimal*, *symmetric*, *scale invariance*, and *individually monotonic* (Kalai and Smorodinsky, 1975).

This monotonicity property can be generalized to arbitrarily n in several ways. When comprehensiveness is dropped, a difficulty with a straightforward extension is noted by Roth (1979d), but recall that we are imposing this condition in our exposition here. One possibility is the following. Fix $i \in N$. Suppose that the feasible set expands but that the set of vectors attainable by the other agents when agent i's payoff is his disagreement payoff remains the same. This implies that for each such vector, the maximal payoff attainable by agent i stays the same or increases; then agent i should not be hurt. The Kalai-Smorodinsky characterization does not extend to arbitrarily many agents if this formulation is adopted. A related property is **restricted monotonicity**, which says that if a feasible set expands without the ideal point being affected, then no agent should be hurt (also, see Rosenthal, 1976). This version is not quite as compelling but together with *weak Pareto-optimality*, *symmetry*, *scale invariance*, and *continuity*, it does lead to a characterization of the Kalai-Smorodinsky solution in general (Segal, 1980). Note that *Pareto-optimality* itself is not achieved. Other formulations are explored in Thomson (1980). Characterizations of a lexicographic extension of the Kalai-Smorodinsky solution that does achieve *Pareto optimality* have been based on variants of *individual monotonicity* (Imai, 1983; Chang and Liang, 1998). For two agents, variants of the Kalai-Smorodinsky solution are defined and characterized by Salonen (1985, 1987), and weighted lexicographic Kalai-Smorodinsky solutions by Dubra (2001). The solutions that satisfy the Kalai-Smorodinsky axioms except for *symmetry* are described by Peters and Tijs (19884, 1985). A characterization of the lexicographic equal loss solution (Chun, 1991) along similar lines is also available.

Generalizing some of the previous ideas, **replacement** (Chun, 2005a) says that if the feasible set changes, but its intersection with the coordinate subspace pertaining to a group of agents remains the same, then these agents' payoffs should all be affected in the same direction. The egalitarian solution is the only solution satisfying *weak Pareto-optimality*, *symmetry*, *continuity*, *contraction independence*, and *replacement*. A related condition leads to a characterization of the Kalai-Smorodinsky solution.

Strong monotonicity says that an expansion of the feasible set should not hurt any agent. The egalitarian solution is the only one to be *weakly Pareto-optimal*, *symmetric*, and *strongly monotonic* (Kalai, 1977b). When comprehensiveness of problems is not imposed, the implications of this requirement in the absence of *weak Pareto optimality* are described by Roth (1979a). Dropping *symmetry*, we obtain the family of monotone path solutions (Thomson and Myerson, 1980). A characterization of the lexicographic egalitarian solution is obtained to Chun and Peters (1988) by insisting on *Pareto optimality* but working with a conditional version of *strong monotonicity*, in which the domination of the solution outcomes in the conclusion of the axiom is not imposed if an increase in an agent's utility can be achieved without the other agent being hurt. Other characterizations of lexicographic solutions are available in which a monotonicity requirement is central (Chang and Hwang, 1999, 2001; Chen, 2000). Characterizations of the equal loss solutions and of variants designed to recover *individual rationality* and *Pareto optimality* are proposed by Chun and Peters (1991), and Herrero and Marco (1993).

Domination is the strongest solidarity requirement: it says that irrespective of the geometric relation between two problems, the solution outcome of one of them should dominate the solution outcome of the other (Thomson and Myerson, 1980). It is satisfied by the monotone path solutions.

In each of the monotonicity properties discussed so far, the change in the geometry of the problem is evaluated before knowing the solution outcome of the initial problem. It seems most natural to take that information into account however. Here is a series of properties achieving this objective for the two-agent case (Thomson and Myerson, 1980). **Adding** says that if a problem is augmented by the addition of alternatives that are all at least as good for agent i as the initial solution outcome and at most as good for agent j, then agent i should not lose and agent j should not gain. The conclusions of the next two axioms are the same. **Cutting** pertains to the subtraction of alternatives that are all at most as good for agent i as the initial solution outcome and at least as good for agent j. **Twisting** combines the previous two operations: it pertains to the simultaneous (i) addition of alternatives that are all at least as good for agent i as the initial solution outcome and at most as good for agent j, and (ii) the subtraction of alternatives that are all at most as good for agent i as the initial solution outcome and at least as good for agent j. These axioms are satisfied quite generally.

The case of more than two claimants is more complex; these properties can be generalized in a number of ways.

• Decomposability. Suppose that after the solution outcome of some problem is calculated, the feasible set expands. There are two ways to look at this new situation: one is to ignore the initial solution outcome—let us call it x—and apply the solution directly to the larger feasible set; the other is to add to x the solution outcome of the larger feasible set, using x as disagreement outcome. When imposed in conjunction with *weak Pareto-optimality* and *symmetry*, an axiom of **decomposability**, which says that both perspectives should produce the same payoff vector, leads to the weighted egalitarian solutions (Kalai, 1977). A related formulation is proposed by Myerson (1977), which also delivers the egalitarian solution. A conditional version of the requirement, obtained by adding the hypothesis that the vectors of utility gains be proportional, is satisfied by the Nash solution, and a characterization of this solution based on this axiom can be developed (Chun, 1988). A different notion of decomposability is formulated by Ponsatí and Watson (1998), who base on it characterizations of the Nash and utilitarian solutions. Ehtamo and Ruusunen (1993) express a related idea, and show its equivalence to *contraction independence*.

• Additivity and multiplicability. Consider opportunities simultaneously coming in two separate feasible sets (the disagreement point being the same for both). One can think of two ways of handling such situations: solving each problem separately and adding the results, or adding the problems and solving the sum problem. **Super-additivity** says that the second way of proceeding should Pareto-dominate the first way, thereby guaranteeing that all agents will agree on the best course of action. For two agents, the Perles-Maschler solution is the only one to be *Pareto-optimal*, *symmetric*, *scale invariant*, *super-additive*, as well as *continuous* on the subclass of strictly comprehensive problems (Perles and Maschler, 1981). If *symmetry* is not imposed, a two-parameter family of solutions emerges (Perles, forthcoming-a). The role of *continuity* is also well-understood (Perles, forthcoming-b). For three agents, the Perles-Maschler axioms are generally incompatible (Perles, 1982), but a class of problems on which compatibility is recovered is identified by Pallaschke and Rosenmüller (2007).

Peters (1986a) proposes a weak form of *super-additivity* that is satisfied by the weighted Nash solutions and characterizes the family they constitute by imposing it in addition to *Pareto optimality*, *strong individual rationality*, *scale invariance*, and *continuity*. For an alternative weakening, he obtains a characterization of the weighted egalitarian solutions however.

Super-additivity is closely related to an axiom motivated by situations in which the feasible set may not be known with certainty. Suppose that it may be either one of two sets. By making contingent agreements today, Pareto-improvements can be achieved over waiting until uncertainly is resolved. **Concavity** says that the expected value of the solution outcomes selected then should Pareto-dominate the solution outcome of the expected feasible set. For *scale invariant* solutions, *concavity* and *super-additivity* are equivalent. **Linearity** says that the solution outcome should be a linear function of the feasible set. On the class of fully comprehensive problems egalitarian and utilitarian solutions are the only solutions satisfying *weak Pareto-optimality*, *symmetry*, *translation invariance contraction independence*, and *concavity*. Also, on that class, the utilitarian solutions are the only solutions satisfying *Pareto-optimality*, *symmetry*, and *linearity* (Myerson, 1981; in these statements, the utilitarian solutions are included if appropriate tie-breaking rules are applied.) For two alternative ways of weakening *linearity*, *scale invariance* is recovered, and the Nash solution emerges. **Conditional linearity 1** is obtained from *linearity* by adding the hypothesis that the two solution outcomes be the same. The Nash solution is the only one to satisfy *Pareto-optimality*, *symmetry*, *scale invariance*, and *conditional linearity 1* (Chun, 1988). **Conditional linearity 2** is obtained from *linearity* by requiring that the boundaries of the component problems be smooth at their solution outcomes and that the sum of these outcomes be Pareto optimal for the sum problem. For two agents, the weighted Nash solutions are the only solutions to satisfy *Pareto-optimality*, *strong individual rationality*, *scale invariance*, *continuity*, and *conditional linearity 2* (Peters, 1986a).

Super-additivity involves the addition of problems. “Multiplications” of problems can also be defined: given two payoff vectors x and y, their product is the vector $(x_1y_1, \ldots, x_ny_n)$. The product of two sets is obtained as the union of all the points obtained in this manner, when x and y are chosen arbitrarily in each of the two sets. **Multiplicability** says that if the product of two sets is a well-defined problem (convexity is not preserved under this operation, hence the proviso), then its solution outcome should be the product of the solution outcomes of the component problems. Together with *Pareto-optimality* and *symmetry* (note that *scale invariance* is not imposed, being a consequence of *multiplicability*), only one solution is admissible, the Nash solution (Binmore, 1984).

When agents may face one of several problems and evaluate their prospects according to the maximin criterion, a natural efficiency condition is met only by the monotone path solutions, the maximax criterion leading to the dictatorial solutions (Bossert, Nosal and Sadanand, 1996). In the two-agent case, minimax regret delivers a counterpart for the equal-loss solution of the monotone path solutions (Bossert and Peters, 2001). When it is the disagreement point that may take several values, the maximin criterion leads to generalizations of the monotone path solutions (Bossert and Peters, 2002). These authors examine the implications of other criteria. A related contribution is Bossert and Peters (2000).

• Invariance with respect to operations performed on utilities. *Scale invariance* can be decomposed into two more elementary invariance properties: **Translation invariance** states that the addition of constants to utility functions should be accompanied by the corresponding translation of the solution outcome. The other part states that the multiplication of the utility functions by positive scalars should be accompanied by a similar operation on the coordinates of the solution outcome. **Homogeneity** is the more limited version of this property when the multiplicative coefficients are the same for all agents.

Ordinal invariance is the requirement that the solution should only depend on the ranking of alternatives implicit in the utility scales. (Convexity is not preserved under these transformations, so this property is studied on domains of possibly non-convex problems.) For two agents, no solution is *strongly individually rational* and *ordinally invariant* (Shapley, 1969; Roth, 1979). However, for three agents, a *Pareto-optimal* solution satisfying both properties can be constructed (Shubik, 1982). There are others, in particular on the class of smooth problems (Shapley, 1984). Sprumont (2000) identifies a family of problems having the property that an arbitrary problem can be obtained from only one member of the family by subjecting it to an ordinal transformation, and this transformation is unique. Thus, this family can be thought of as an **ordinal basis** of the space of problems. To define an *ordinally invariant* solution, it then suffices to specify how the solution behaves on this basis. Such a solution can be constructed for the n-agent case inspired by the Shubik solution (Safra and Samet, 2004). Another solution is proposed by Samet and Safra (2005), whose definition exploits a technique developed by O'Neill, Samet, Wiener, and Winter (2004) to deal with a generalization of Nash's model (Section 10). Other contributions to the understanding of *ordinal invariance* are by Kıbrıs (2002b, 2003, 2004) and Sákovics (2004). Calvo and Peters (2005) impose *ordinal invariance* for some players and *scale invariance* for others.

Weak ordinal invariance (Nielsen, 1983) requires invariance under the application of the *same* increasing transformation to all utilities. In the two-agent case, on the subclass of problems whose Pareto-optimal boundary is connected, the lexicographic egalitarian solution is the only solution satisfying *Pareto-optimality*, *strong individual rationality*, *weak ordinal invariance*, and *contraction independence* (Nielsen, 1983; 1984).

8 Relational axioms pertaining to the disagreement point

• Monotonicity. The disagreement point represents the agents' fall-back positions. Thus, it is natural to to require that if an agent's disagreement utility increases, he should not lose. This is the property of **disagreement point monotonicity** (Thomson, 1987a). A related property, **strong disagreement point monotonicity** (Thomson, 1987a) says that such a change should benefit none of the other agents. *Disagreement point monotonicity* is met by most solutions, but its strong form is quite demanding: the egalitarian solution and variants do satisfy it, but none of the other central solutions does. Engwerda and Douven (2008) formulate a local version of these properties, and identify classes of problems under which the Nash solution satisfies them. A related requirement, that a simultaneous increase in an agent's disagreement utility and decrease in some other agent's disagreement utility should not hurt the first agent and not benefit the second agent (Thomson, 1987), is the counterpart of the requirement that there should be "no transfer problem" discussed in the theory of international trade. It is studied by Bossert (1994a) together with the requirement that the impact of such a change on the other players should be bounded by the impact it has on the agents whose disagreement utilities change. Other relevant contributions on this issue are by Furth (1990) and Chun (1996).

Ideal point monotonicity says that if the disagreement point changes in such a way that at the resulting ideal point, agent i's payoff is greater and the payoffs of all other agents remain the same, then agent i should not lose. On the class of two-agent fully comprehensive problems, the equal-loss solution is the only solution to satisfy *weak Pareto-optimality*, *symmetry*, *translation invariance*, *restricted monotonicity*, *ideal point monotonicity*, and *continuity* (Chun, 1988a).

• Axioms pertaining to the shapes of inverse sets. Other properties pertaining to situations in which there may be uncertainty in the disagreement point have been formulated. They can be described by means of geometric properties of the set of disagreement points from which a particular alternative is recommended as solution outcome for each particular feasible set S. Given a point x in S, the **inverse set of the solution for S and x** is the set of disagreement points from which the solution leads to x. A solution has **convex inverse sets** if, whenever the same point—let us call it x—is the solution outcome when either d^1 or d^2 is the disagreement point, then x is the solution outcome when the disagreement point is any convex combination of d^1 and d^2. It has **star-shaped inverse sets** if, denoting x the solution outcome when the disagreement point is d, x is the solution outcome when the disagreement is any convex combination of x and d. It has **cone-shaped inverse sets** if, denoting x the solution outcome when the disagreement point is d, x is the solution outcome when the disagreement point is any point on the half-line passing through x and d and having x as endpoint. These properties appear in characterizations of the Nash solution and of the Kalai-Rosenthal solution (Kalai and Rosenthal, 1978; this solution is a variant of the Kalai-Smorodinsky solution) due to Chun (1990), and de Clippel (2007). They are also discussed in Furth (1990) and Peters and van Damme (1991) who offer characterizations of the Nash and continuous Raiffa solutions based on them. Dagan, Volij, and Winter (2002) offer a variant.

• Operations performed on disagreement points. **Disagreement point**

concavity is a counterpart of our earlier concavity requirement pertaining to feasible sets. On the domain of fully comprehensive problems, the weighted egalitarian solutions are the only solutions to satisfy *weak Pareto-optimality*, *independence of non-individually rational alternatives*, *continuity*, and *disagreement point concavity* (Chun and Thomson, 1990a; Salonen, 1996, proves a variant of these results). Generalizations of the egalitarian solution called **directional solutions** are obtained if the independence requirement is weakened to *individual rationality* (Chun and Thomson, 1990a). A characterization of the lexicographic egalitarian solution along similar lines is established by Chun (1989).

Weak disagreement point concavity says the following: consider two problems with the same feasible set. If the boundary of this set is smooth at their solution outcomes, and the weighted average of these points with weights λ and $1-\lambda$ is Pareto-optimal, then for the problem in which the disagreement point is the weighted average of the disagreement points of the two problems with weights λ and $1-\lambda$, this point should be the solution outcome of the problem whose disagreement point is this weighted average. Weakening *disagreement point concavity* in this manner allows recovering *scale invariance*, but the Nash solution is the only solution to satisfy *Pareto-optimality*, *symmetry*, *weak disagreement point concavity*, and *continuity* (Chun and Thomson, 1990c; related results are due to Peters and van Damme, 1991, and Salonen, 1998).

9 Relational axioms pertaining to the population of agents

Next, we turn to situations in which the population of agents involved may vary. To study the issue, a generalization of the model is required. We imagine an infinite set of "potential" agents, indexed by the natural numbers $\mathbb{N}$, but finitely many of them are involved in each particular problem. Here, a solution associates with each finite set of agents and each of the problems that they could face, a payoff vector for that problem. Examples are the solution obtained by always applying the Nash formula, and the solution obtained by always applying the Kalai-Smorodinsky formula. A third one is obtained by applying the Nash formula when the number of agents is even and the Kalai-Smorodinsky formula otherwise. However, one expects the resolutions of problems involving different populations of agents to be somehow related, and several criteria have been formulated for that purpose. An account of the theory that has been elaborated to deal with variations in populations is Thomson and Lensberg (1989). Two axioms have been central, a monotonicity axiom and an independence axiom.

• Population monotonicity. First, starting from some initial problem for which a solution has made a particular recommendation, imagine that some agents leave, relinquishing their rights. **Population monotonicity**[5] says that none of the remaining agents should lose. If one thinks of the problem of dividing a social endowment of resources among a group of agents, saying that some agents are not present is the same thing as saying that they are present but giving them nothing. Thus, and working for simplicity with utility functions that assign 0-utility to the 0-bundle, the intersection of the image in utility space of the problem involving the larger population with the coordinate subspace pertaining to the smaller group is the image in utility space of the feasible set for the smaller group. The Nash solution violates *population monotonicity* but both the Kalai-Smorodinsky and egalitarian solution satisfy it and characterizations of these solutions in which the axiom is central are available: the Kalai-Smorodinsky solution is the only solution to satisfy *weak Pareto optimality*, *anonymity*, *scale invariance*, *population monotonicity*, and *continuity* (Thomson, 1983a); also, the egalitarian solution is the only one to satisfy *weak Pareto optimality*, *symmetry*, *contraction independence*, *population monotonicity*, and *continuity* (Thomson, 1983b; variants of this characterization are Thomson, 1984a,b).

Conversely, when agents come in, one may be interested in measuring the impact this event has on each of the agents initially present. The ratio of his final payoff to initial payoff can be used for that purpose. Seen positively, the minimal value taken by this ratio can be interpreted as a guarantee offered to the agent that his final utility will be at least a certain fraction of his initial utility. The collection of these minimal values indexed by pairs of initial and final populations is the **guarantee structure** of the solution. Solutions that offer greater guarantees are more desirable. It so happens that maximal elements within broad families of solutions satisfying particular lists of basic properties can be identified. Similar definitions can be given pertaining to groups of agents, resulting in the notion of a **collective**

[5]A general presentation of the various applications of the idea of population monotonicity is Thomson (1995a).

guarantee structure of a solution (Thomson and Lensberg, 1983; Thomson, 1983a; Chun and Thomson, 1989). Parallel analysis can be performed of the opportunities for gains that solutions provide to agents when population varies, giving rise to concepts of **opportunity structures**, and **collective opportunity structures** (the definitions can indeed be adapted for groups of agents) (Thomson, 1987b). Depending upon whether the focus is on guarantees or opportunities, for individuals or for groups , the Kalai-Smorodinsky solution or the egalitarian solution, and to a lesser degree, the Nash solution, are found to be maximal elements within broad classes of solutions.

• Consistency and its converse. After applying a solution to some problem, let us imagine that some agents "leave" with their components of the payoff vector x it selects and let us assess the opportunities available to the agents who remain. They consist of all the payoff vectors in the initial problem at which the departing agents receive their components of x. Geometrically, this **reduced problem** is a slice of the original problem parallel to the subspace pertaining to the remaining agents that passes through x. **Consistency** says that its solution outcome should be the restriction of x to the set of remaining agents. The Nash solution satisfies this requirement but neither the Kalai-Smorodinsky solution nor the egalitarian solution does. In fact, the Nash solution is the only solution to satisfy *Pareto-optimality*, *anonymity*, *scale invariance*, and *consistency* (Lensberg, 1988; variants of this characterization are due to Thomson, 1985b, and Lensberg and Thomson, 1988).[6]

Weak consistency is the property obtained from *consistency* by requiring that in the reduced problem, the payoff to each remaining agent should be at least as large as it was in the initial problem (instead of requiring equality). The two axioms are conceptually close, and so are they mathematically since on the subclass of strictly comprehensive problems, they are equivalent. The egalitarian solution is the only one to satisfy *weak Pareto optimality*, *symmetry*, *continuity* and the two variable population properties of *population monotonicity* and *weak consistency* (Thomson, 1984c).

A characterization of the lexicographic egalitarian solution is obtained by dropping *scale invariance*, adding a feasible set monotonicity property and **weak continuity**, a continuity requirement based on a notion of convergence that takes into account not only how sequences of problems converge but also

[6]A general presentation of the various applications of the "consistency principle" is Thomson (2007b).

how associated sequences of reduced problems converge (Lensberg, 1985a; Thomson and Lensberg, 1989). A parameterized notion of a property related to *consistency*—it applies only to the departure of the agents whose payoffs are the lowest—is proposed by Blackorby, Bossert, and Donaldson (1996). This notion appears in a characterization of a subfamily of the generalized Gini solutions that they develop (the coefficients defining the components of the solutions for different populations have to be related). An alternative notion of consistency involving ideal points is proposed by Peters, Tijs and Zarzuelo (1994). They base on it a characterization of the Kalai-Smorodinsky solution.

Next, we define a new family of solutions. To each potential agent $i \in \mathbb{N}$ we attach a continuous and increasing function $f_i \colon \mathbb{R}_+ \to \mathbb{R}$. To the list $f \equiv (f_i)_{i \in \mathbb{N}}$, we then associate the solution that selects, for each problem (S, d) with agent set $N \subset \mathbb{N}$ such that $d = 0$, the maximizer with respect to $x \in S$ of the sum $\sum_{i \in N} f_i(x_i)$. If $d \neq 0$, we simply translate the problem so that this be the case before performing the maximization, and we carry out the inverse translation to obtain the solution outcome of the initial problem. (The list f should satisfy an additional minor requirement for the maximizer to be unique.) We thereby obtain the **separable additive solution associated with $\boldsymbol{f}$**: these solutions are the only solutions satisfying *weak Pareto-optimality*, *continuity*, and *consistency* (Lensberg, 1987).

Now, consider a problem and an outcome x as a possible solution outcome. Let us check whether for each two-agent subgroup of the agents involved in the problem, the restriction of x to this subgroup is the solution outcome of the associated reduced problem they face (this is the problem obtained by imagining that all but these two agents leave with their components of x). **Converse consistency** says that if the answer is yes, then x should be the solution outcome of the initial problem. This axiom is violated by the Nash solution on our primary domain, but it is satisfied on the class of smooth problems. It is violated by the Kalai-Smorodinsky solution and satisfied by the egalitarian solution. Studies of its implications in conjunction with other standard properties are due to Chun (2000, 2002), who offers characterizations of the Nash and egalitarian solutions involving it.

Given two problems, suppose that (i) for each member of a particular group of agents, the solution recommends the same payoffs in both, and (ii) the sets of payoff vectors at which these agents receive these payoffs are the same in both problems. Then **separability** (Chun, 2005) requires that the other agents should also receive the same payoffs in the two problems.

This requirement is implied by *contraction independence*, and even though it is a fixed population property, it is closely related to *consistency*. The Nash solution is the only solution to be *Pareto optimal*, *anonymous*, *scale invariant*, *continuous*, and *separable*. A characterization of the egalitarian solution is obtained on the basis of *weak Pareto optimality*, *symmetry*, *continuity*, *converse consistency*, and a condition of **domination-separability**, obtained from *separability* by dropping (ii) from the hypothesis and only requiring a domination relation between the two solution outcomes (without the direction of the inequality being specified).

Instead of allowing for general augmentations into spaces of higher cardinalities, a more limited formulation is when problems are replicated (Kalai, 1977a; Thomson, 1986). Yet another geometric relation between problems is considered by Chae and Heidhues (2004).

10 Variants and enrichments of the model

The directions in which the literature on the bargaining problem has evolved following Nash's paper can be sorted according to the extent to which they question the basic model.

10.1 Variants of the model.

First are reformulations that retain the components of what defines a bargaining problem or a solution, but the specific properties required of them are relaxed or modified.

• The feasible is not required to be convex (Myerson, 1977; Kaneko, 1980; Herrero, 1989; Anant, Basu, and Mukherji, 1990; Conley and Wilkie, 1991, 1996a, 1996b; Zhou, 1996; Serrano and Shimomura, 1998; Ok and Zhou, 1999; Denicolò and Mariotti, 2000; Mariotti, 2000c; Hougaard and Tvede, 2003; Peters and Vermeulen, 2007; Xu and Yoshihara, 2007; Lombardi and Mariotti, 2008).

• Further, the feasible set is a finite set (Mariotti, 1988; Anbarci, 2006; Nagahisa and Tanaka, 2002; Kıbrıs and Sertel, 2007; Peters and Vermeulen, 2007).

• Solutions are allowed to be multi-valued (Thomson, 1981a; Peters, Tijs and Koster, 1983; Herrero, 1989; Blackorby, Bossert, and Donaldson, 1994; Peters and Tijs, 1984; Serrano and Shimomura, 1998; Ok and Zhou, 2000; Peters and Vermeulen, 2007; Hinojosa, Mármol, and Zarzuelo, 2008; Lombardi and Mariotti, 2008).

• Solutions select probability distributions over outcomes (Peters and Tijs, 1984).

10.2 Related models formulated in utility space.

The basic model can also be enriched by the addition of other data.

• A point inside the feasible set is added. This point can be interpreted as a first-round compromise, a settlement reached on a previous occasion when opportunities were more limited, or as an exogenous lower bound on payoffs (Gupta and Livne, 1988).

• A point to the northeast of the Pareto boundary is added. It may represent claims that agents have, promises that may have been made to them, or the settlement reached on a previous occasion, when opportunities were better and on which their expectations may depend, or an exogenous upper bound on payoffs. We then obtain a **bargaining problem with claims** (Chun and Thomson, 1992). This model (Bossert, 1992; 1993; Gerber, 1997, 2005; Herrero, 1998, 1995; Gächter and Riedl, 2005; Lombardi and Yoshihara, 2008) can also be thought of as a non-transferable utility generalization of the model of bankruptcy (O'Neill, 1982; for a survey, see Thomson, 2003). The special case when an extra point is specified on the boundary has also been considered (Brito, Buoncristiani and Intriligator, 1977). A model in which the disagreement point is replaced by a claims point is studied by Mariotti and Villar (2005).

• A reference point is added to the feasible set, and this point may or may not be feasible (Rubinstein and Zhou, 1999; Pfingsten and Wagoner, 2003).

• The disagreement point is dropped altogether. We then obtain what is sometimes called a "social choice problem" (Harsanyi, 1955; Thomson, 1981c; Klemisch-Ahlert, 1991; Conley McLean, and Wilkie, 1997; Vartiainen, 2007).

• A list of disagreement points is specified, indexed by agents, each of these points being interpreted as the payoff vector that obtains if the corresponding agent is responsible for the breakdown of negotiations. One would indeed expect the outcome eventually chosen to depend on who that agent is (Kıbrıs and Gürsel, 2005a, b).

• The disagreement point is replaced by a set of disagreement points, no probabilistic structure being placed on this set (Basu, 1996).

• The disagreement point is random and its probability distribution is known (Smorodinsky, 2005; Livne, 1988).

• The disagreement point is written as a function of the feasible set (Esteban and Sákovics, 2007).

• The feasible set evolves over time (Livne, 1987; O'Neill, Samet, Wiener, and Winter, 2003).

• Information is added about what groups of agents can achieve: Nash's formulation only specifies a feasible set for the entire set of agents, and a payoff that each agent can guarantee himself. These are the feasible sets for the "grand coalition" and the individual agents. In richer multi-agent interactions, other groups may also be able to achieve something. How to take into account the opportunities open to all groups is the subject of the theory of coalitional games.[7] A number of authors have been particularly interested in developing solutions to this class of games that extend solutions to bargaining problems (Harsanyi, 1959, 1963).

• A higher-level perspective can be taken, and one can imagine that agents bargain not so much about specific payoff profiles but rather about solutions themselves. Then the model would include their preferences over solutions (Border and Segal, 1997; Segal, 2000; Pivato, 2008).

10.3 Adding information about physical outcomes

A more radical reformulation is obtained by including information about the set of underlying *physical outcomes*. Such information is ignored in Nash's **welfarist** formulation: two conflict situations may have the same representation in utility space and yet differ in ways that intuitively one feels matters. Economic conflicts are typically described with a variety of additional detail that could and perhaps should be taken into account. The spaces of feasible alternatives (they are allocations then) have convex, order, and topological structures that suggest restrictions on preferences and properties of allocation rules that are not meaningful otherwise, and allow constructing rules that would not be well-defined for a model formulated in utility space.

• Responsiveness to risk. A question that can be addressed in this framework is how solutions respond to changes in the agents' risk aversion. Is it preferable to face a more risk-averse opponent? To study this issue we need to explicitly introduce the set of underlying physical alternatives (for

[7]For a treatment of the theory, see Peleg and Sudhölter (2007).

background references on concepts of risk aversion, see Pratt, 1964, Yaari, 1969). An ***n*-person concrete problem** is a list (C, e, u), where C is a set of **certain options**, $e \in C$, and $u = (u_1, \ldots, u_n)$ is a list of von Neumann-Morgenstern utility functions defined over C. The **abstract problem associated with (C, e, u)** is the pair $(S, d) \equiv (\{u(\ell)|\ell \in L\}, u(e))$.

The first property we formulate focuses on the agent whose risk-aversion changes. According to his old preferences, does he necessarily lose when his risk-aversion increases? If yes, the solution is **risk-sensitive**. **Strong risk-sensitivity** focuses on the agents whose preferences are kept fixed. It says that all of them should benefit from the increase in some agent's risk-aversion.

A concrete problem is **basic** if each of the Pareto-optimal alternatives of its associated abstract bargaining problem is the image of a sure physical alternative. If a problem is basic and agent i's utility is replaced by a more risk-averse utility, then the new problem still is basic. On the domain of basic problems, the Nash solution is *risk-sensitive* but not *strongly risk-sensitive.* On this domain, the Kalai-Smorodinsky solution is *strongly risk sensitive* (Kihlstrom, Roth and Schmeidler 1981, Nielsen 1984).

The following are interesting logical relations. If a solution is *Pareto-optimal* and *risk sensitive*, then it is *scale invariant.* If a solution is *Pareto-optimality* and *strongly risk sensitive*, then it is *scale invariant* (Kihlstrom, Roth, and Schmeidler 1981). For two agents, additional relations also exist between *risk sensitivity* and *twisting* (Tijs and Peters 1985) and between *risk sensitivity* and *midpoint domination* (Sobel 1981). Further results appear in Wakker, Peters, and van Riel (186), Koster, Peters, Tijs and Wakker (1983), Peters (1987a), Peters and Tijs (1981, 1983, 1985a), Tijs and Peters (1985), and Klemisch-Ahlert (1992a).

For non-basic problems, two cases should be distinguished. If the disagreement point is the image of one of the basic alternatives, what matters is whether the solution is appropriately responsive to changes in the disagreement point (Section 8). Suppose $C = \{c_1, c_2, e\}$ and let F be a solution on Σ_d^2 satisfying *Pareto-optimality*, *scale invariance*, and *disagreement point monotonicity.* Then, if u_i become more risk-averse, agent j gains if $u_i(\ell) \geqq \min\{u_i(c_1), u_i(c_2)\}$ and not otherwise (based on Roth and Rothblum 1982 and Thomson 1987a)

The n-agent case is studied by Roth (1988). Situations when the disagreement point is obtained as a lottery are considered by Safra, Zhou, and Zilcha (1990). For a model based on Yaari's dual theory of choice under risk, see

Volij and Winter (2002), and for the case when agents have rank-dependent utilities, see Denicolò (2000) and Köbberling and Peters (2003). An application to insurance contracts is discussed in Kihlstrom and Roth (1982), and to the reallocation of individual endowments by Klemisch-Ahlert (1992). How an agent ranks his opponents in terms of their risk aversion is studied in a special case by Cressman and Gallego (2009).

• Resource allocation. Our canonical allocation rule in economics, the Walrasian equilibrium, cannot be discussed without information about commodities, endowments, and preferences over commodity bundles. In the last ten years, the axiomatic program has considerably developed so as to accommodate this supplementary information. One may still work with solutions defined on images in utility space of some underlying set of physical alternatives, but consider changes in the components a problem that are due to changes in the underlying data. For instance, a counterpart of *strong monotonicity* is obtained by requiring that if the physical resources available increase, then no agents' payoff should decrease. Chun and Thomson (1988) study this and related properties and show that the number of goods is critical for solutions to respond appropriately. Although these properties are often satisfied in the one-good case, they generally do not as soon as there are two goods, and of course if there are more than two goods. Whether solutions are immune to the transfer problem or to manipulation by destruction of part of one's endowment is studied by Chun (1996) and Cho and Chun (2000) in economies with or without production. Here also, two is the critical number of goods beyond which violations occur.

Roemer (1986a,b, 1988, 1990) has criticized the welfarist assumption underlying Nash's model, and after transcribing axioms to a concretely specified setting of resource allocation, has patterned characterizations of all of the solutions that have been central in the theory. His formulation includes utilities over commodity bundles, and a important axiom in his work is a requirement of invariance with respect to variations in the number of goods. See also Nieto (1992), Iturbe-Ormaetxe and Nieto (1996), Segal (2000), Rotar and Smirnov (1992), and Chen and Maskin (1999), the latter two references including production sets in the model. A model with claims formulated to address similar issues is studied by Herrero (1998).

Experimental work supporting non-welfarist formulations is reported by Roth and Malouf (1979), and Yaari and Bar-Hillel (1984). Further experimental studies are by Roth and Murningham (1980), Felsenthal and Diskin (1982), and Klemisch-Ahlert (1996). An application of bargaining solutions to a labor market problem is studied by Gerber and Upmann (2005) and to a problem of reallocation of endowments by Klemisch-Ahlert and Peters (1994) .

As challenges to the standard von Neumann-Morgenstern theories of decision under uncertainty have mounted in the last few years and alternate theories developed, the need and the possibility of revisiting the bargaining problem has increased. The literature is also developing in this direction. An early contribution to the issue is by Rubinstein, Safra, and Thomson (1992) who propose a definition and a characterization of an ordinal version of the Nash solution in this context. Follow-up papers are by Burgos (1993, 1995), Hanany and Safra (1998), Grant and Kajii (1995), Safra and Zilcha (1993), Valenciano and Zarzuelo (1994, 1997), Houba, Tieman, and Brinksman (1998), Denicolò (2000), Burgos, Grant, and Kajii (2002), Shalev (2002), Zhou (2007), Hanany (2007, 2008), and de Clippel (2008).

Taking a step further, Moulin and Thomson (1998) consider an entirely ordinal formulation, and ask about monotonicity in welfares as a response to an increase in the social endowments of goods. A considerable expansion of the axiomatic program has occurred in the last twenty years. These models bear little relation to Nash's original model but the same general ideas of monotonicity and independence often underlie the central axioms. This program can be titled the "axiomatics of resource allocation".

11 Strategic issues

Many authors have discussed Nash's contributions to strategic analysis, but a few words are necessary here to link Nash's axiomatic and strategic models. A presentation of the basic approaches can be found in Binmore (1987a, b).

11.1 Strategic games of proposals and solutions.

• The strategic game that Nash (1953) proposed to append to his abstract description of the bargaining problem is simple and natural: each agent announces a payoff for himself, and the outcome is the profile of these payoffs if it is feasible, and the disagreement point otherwise. Nash proved that if a small amount of uncertainty is added in the specification of a problem, the Nash equilibrium payoff vector of the game is close to the outcome selected by the Nash bargaining solution. A related formulation by Anbar and Kalai

(1978) also delivers the Nash solution outcome. At an equilibrium of the sequential game proposed by Bossert and Tan (1995), the egalitarian outcome is obtained.

• Instead of allowing any payoff demand, one may want to discipline the process by requiring that demands be rationalizable by some well-behaved solution. One can imagine agents having agreed that a solution in some reasonable class should be used. The class could be defined by enumeration, or by the axioms that all of its members should satisfy. Then, a strategy choice for an agent is a member of that family. A sequential procedure in which the feasible set is repeatedly truncated by eliminating all points at which an agent receives more than the payoff the solution he proposed would assign to him, is defined, and the solutions are reapplied. For a plausible family of admissible solutions, convergence of the two proposals to a single point occur. Thus, the process defines a strategic game in which strategies are bargaining solutions. At an equilibrium, only the Nash cooperative outcome is obtained (van Damme, 1986). An equally plausible reformulation however leads to the Kalai-Smorodinsky solution (Chun, 1984). See also Naeve-Steinweg (1997, 1999). Non-strategic formulations of such recursive processes are studied by Thomson (2008), Anbarci and Yi (1992), Marco, Peris, and Subiza (1995), and Naeve-Steinweg (2004).

• Stahl (1977) and Rubinstein (1982) formulate a sequential game in which agents take turns making offers, and calculate their equilibrium outcomes. They too obtain the Nash outcome at the limit. Precursors of the approach are Zeuthen (1930), Harsanyi (1956, 1958), and Saraydar (1965). This result is often thought of as another vindication of the Nash solution and in fact, those who do not adhere to the axiomatic approach or do not find Nash's strategic story compelling, may think of it as a more convincing way to justify the solution. Of course, the limit of the equilibrium correspondence of these games satisfy Nash's axioms, and if one is uncomfortable with the axioms, one should be concerned with a model that unwittingly put them in. Moreover, these conclusions turn out not to be as robust as one could have wished, especially for more than two agents. They depend on the particular way in which the non-cooperative game is specified, (who speaks when and says what, and what rules determine termination of negotiations and the final payoffs). Important follow-up contributions are Binmore (1987a,b). Additional papers are by Crawford (1980), Binmore (1987), Anbarci (1989, 1992, 1993, 1997), Peleg (1997), Spinnewijn and Spinnewijn (2007), and Anbarci and Boyd (2008). Calvo and Gutiérrez (1994) is in a somewhat different spirit. Krishna and Serrano (1996) show the relevance of *consistency* to the analysis of certain strategic games.

11.2 Manipulation.

In order to calculate the payoff vector recommended by a solution, we need to know how agents evaluate decisions. By misrepresenting his utility, an agent may be able to enforce an outcome that he prefers to the one that would have been achieved if he had not done so. Of course, the other agents may be in that position too, and the question then is predicting what happens when they all attempt to do so. For that purpose, a manipulation game has to be associated with the solution and its equilibria identified. When the Nash solution is applied to solve one-commodity allocation problems, a dominant strategy for each agent is to announce a linear utility. Then, when all agents play their dominant strategies, equal division results (Crawford and Varian, 1979). The multi-commodity case is more delicate. If preferences are known and agents can only misrepresent their utilities, at equilibrium, each agent announces the least concave numerical function that represents his preferences (Kannai, 1977). If preferences are unknown, and for a number of solutions, Walrasian allocations from equal division for the true preferences are obtained at equilibrium (Sobel, 1981, 2001; Kıbrıs, 2002a). Another contribution to this issue is Gómez (2003).

11.3 Implementation.

Supposing that some solution has been selected as embodying society's objectives, does there exist a game form with the property that for each preference profile, when the game form is played by agents with these preferences, the equilibrium outcomes are the ones selected by the solution for that profile? If yes, the solution is **implementable** (Maskin, 1979; Hurwicz, 1979). Whether a solution is implementable depends on the type of game forms that are used, and on the behavioral assumptions made about how agents confronted to such games behave. For implementation by normal form games and when agents calculate best responses taking as given the choices made by the other agents, a critical property for what is then called **Nash-implementability** is **Maskin-monotonicity** (actually an invariance property with respect to enlargements of lower contour set at the chosen alternative). Most solutions are not *Maskin-monotonic* and therefore not *Nash* *implementable* by normal form games. However, an implementation of the Kalai-Smorodinsky solution by a sequential game is possible (Moulin, 1984). Later contributions delivered the Nash solution (Howard, 1992), the egalitarian solution (Bossert and Tan, 1992), and an extension of the Nash solution to non-convex problems (Conley and Wilkie, 1996b). Serrano (1997b), Trockel (1998, 199a, 2000, 2002a,b, 2003), and Naeve (1999) are other contributions. For subgame perfect implementation, a general theorem is offered by Miyagawa (2002). It covers all solutions obtained, after normalizing problems so that the ideal point has equal coordinates, by maximizing a monotone and quasi-concave function of the agents' payoffs. Implementation is by means of a stage game and the equilibrium notion is subgame perfection. The most recent entry is Vartiainen (2008).

12 Conclusion

Nash's model in his 1950 paper has been an ideal laboratory for the development of the axiomatic program. For this model, a conceptual apparatus and proof techniques were elaborated, and a set of results obtained that are unparalleled in game theory and social choice, except perhaps in the abstract Arrovian theory of social choice. It is of course partly a mathematical accident. Nash's model happens to have the "right richness": It is sufficiently rich to permit the elaboration of a non-trivial theory, and it is not so rich as to be intractable.[8]

It is mainly on the basis of monotonicity properties that the Kalai-Smorodinsky solution should be seen as an important challenger to the Nash solution, the egalitarian solution presenting another appealing choice. This latter solution enjoys even stronger monotonicity requirements and like the Nash solution, it satisfies strong independence conditions. Unlike both the Nash and Kalai-Smorodinsky solutions, it requires interpersonal comparisons of utility however, which, depending upon the context, may be seen as a desirable feature it has, or a limitation.

[8]In particular, it allows a great many *single-valued* solutions to be defined. Almost all other models require that multi-valued be allowed. This almost always lead to complications at both the conceptual level (this forces choices of quantification in the relational axioms that are not always compelling), and the technical level.

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