Rochester Center for
Economic Research

Admissibility and Event-Rationality

Barelli, Paulo and Spyros Galanis

Working Paper No. 568
October 2011

University of
Rochester

Admissibility and Event-Rationality*

Paulo Barelli[†] Spyros Galanis[‡]

October 4, 2011

Abstract

We develop an approach to providing epistemic conditions for admissible behavior in games. Instead of using lexicographic beliefs to capture infinitely less likely conjectures, we postulate that players use tie-breaking sets to help decide among strategies that are outcome-equivalent given their conjectures. A player is event-rational if she best responds to a conjecture and uses a list of subsets of the other players' strategies to break ties among outcome-equivalent strategies. Using type spaces to capture interactive beliefs, we show that common belief of event-rationality (RCBER) implies that players play strategies in $S^\infty W$, that is, admissible strategies that also survive iterated elimination of dominated strategies (Dekel and Fudenberg (1990)). We strengthen standard belief to *validated belief* and we show that event-rationality and common validated belief of event-rationality (RCvBER) implies that players play iterated admissible strategies (IA). We show that in complete, continuous and compact type structures, RCBER and RCvBER are nonempty, and hence we obtain epistemic criteria for $S^\infty W$ and IA.

Keywords: Epistemic game theory; Admissibility; Iterated weak dominance; Common Knowledge; Rationality; Completeness.

*We are grateful to seminar participants at Collegio Carlo Alberto, Rochester, Southampton, Stony Brook, the Fall 2009 Midwest International Economics and Economic Theory Meetings, the Royal Economic Society 2010 Conference, the 10th SAET Conference, the 2010 Workshop on Epistemic Game Theory, Stony Brook, and the Second Brazilian Workshop of the Game Theory Society.

[†]Department of Economics, University of Rochester.

[‡]Economics Division, University of Southampton.

1 Introduction

As noted by Samuelson (1992) and many others, there is a potential problem in dealing with common knowledge of admissibility in games, which is known as the inclusion-exclusion problem. The reason is that a strategy is admissible if and only if it is a best response to a conjecture with full support. If we capture knowledge by the support of the agent's belief and assume that she is rational, that is, she optimizes given her belief, then playing an admissible strategy implies that she must necessarily consider all strategies of the other players as possible, including the strategies that are not admissible. So she cannot believe that her opponents play admissible strategies because she cannot exclude from consideration their inadmissible strategies.

Recently, Brandenburger et al. (2008), henceforth BFK, provided a way of dealing with the inclusion-exclusion issue, by using lexicographic probability systems (LPS) and the notion of *assumption* in the place of *certainty*. Roughly speaking, a player with conjectures that form an LPS can have a fully supported conjecture while "assuming" certain events that are not equal to the whole state space. BFK show that common assumption of admissibility (RCAR) characterizes iterated admissibility (IA), but RCAR is empty in complete and continuous type structures. So BFK do not provide an epistemic characterization of IA. Keisler and Lee (2011) and Yang (2009) have recently extended BFK's analysis and obtained nonemptiness of RCAR. The former allows for discontinuous type mappings, and the latter uses a weaker notion of assumption.

We propose an alternative route. Instead of an LPS-based analysis, we use event-rationality to allow for players to break ties with lists of subsets of opponents' strategies. That is, we use a different notion of rationality: the LPS-based approaches assume that players are lexicographic expected utility maximizers. We assume that players are event-rational. The two notions of rationality equally describe admissible behavior. The difference comes into play in the analysis of interactive beliefs. Interactive beliefs are described by type spaces. In our framework, a type of a player determines her beliefs over the strategies and types of the other players (as in the standard framework) and in addition it determines the tie-breaking list that the (event-rational) type uses. As a result, common belief of event-rationality does not run into the tension of having to exclude and include the same event. In contrast, in an LPS-based analysis a type of a player determines her lexicographic beliefs over the strategies and types of the other players, and the inclusion-exclusion tension is avoided by the use of "assumption" in the place of certainty. Under our approach, we provide epistemic foundations for both the solution concept proposed by Dekel and Fudenberg (1990) ($S^{\infty}W$) and iterated admissibility (IA).

We consider finite two-player games in strategic form. The two players are Ann and Bob, denoted by superscripts "a" and "b". In order to provide some intuition about event-rationality, note that if a strategy s^a of Ann's is rational then it is a best response to some conjecture, $v \in \Delta(S^b)$, where S^b is the set of Bob's strategies. If s^a is inadmissible and therefore weakly dominated by some (mixed) strategy σ^a, then s^a and σ^a give the same payoff for all strategies of Bob on the support of v while σ^a is strictly better than s^a for all conjectures with support on the complement of the support of v. Hence, whenever Ann chooses an admissible strategy, it is as if she optimizes given her conjecture, as usual, but when she is totally indifferent between two strategies she compares them using a measure with support on the difference between S^b and the support of her conjecture. We say that she "breaks ties" using the event that is the complement of her support (with respect to S^b). In other words, Ann is confident in trusting her belief, just like any other rational agent. But if two of her strategies are outcome-equivalent under her belief, she chooses the one that is also optimal under a measure with support being the complement of the support of her belief.

There is nothing particular about breaking ties with respect to the complement of her support when defining event-rationality. Ann can conceivably break ties using any other set, as long as it is outside her current frame, that is, disjoint from the support of her belief.[1] Furthermore, Ann need not use a single such tie-breaking set. She may well have many such sets, each providing extra validation for her chosen strategy.

The principle behind event-rationality is, therefore, the following: if two strategies are outcome-equivalent given Ann's conjecture, then Ann has no way of deciding among them within her frame of mind: the two strategies yield the same outcome for whichever strategy of Bob she considers possible. Ann must, therefore, resort to information beyond her frame to make a decision. She could, for instance, flip coins, that is, resort to fully external means. But in doing so Ann would be neglecting information about her two strategies, contained in

[1]But note that, for the purpose of breaking ties, it suffices to consider only subsets of Bob's strategies. In particular, when we introduce the formal model of interactive beliefs, it is without loss to assume that Ann uses only lists of Bob's strategies to break ties, because lists that include the types of Bob only matter for breaking ties through the strategies of Bob that they are related to.

how they fare against strategies of Bob that are considered impossible by Ann's conjecture. Event-rationality postulates that Ann does not neglect this information. Moreover, in doing so, she does not change what she thinks about Bob's choices.

Turn now to interactive beliefs, captured by type structures. Let T^a and T^b be the sets of types of Ann and Bob. A type $t^a \in T^a$ determines Ann's conjectures over Bob's choices, Ann's beliefs over Bob's types and so on, together with the tie-breaking list used by Ann. A state for Ann is a strategy-type pair (s^a, t^a) and her beliefs over Bob are given by her beliefs over $S^b \times T^b$. A strategy-type pair (s^a, t^a) of Ann's is called event-rational if s^a is optimal given t^a's conjecture and breaks ties for all sets in t^a's tie-breaking list. Event-rationality and common belief of event-rationality is then captured as the intersection of infinitely many events: Ann is event-rational, and so is Bob; Ann is certain that Bob is event-rational and Bob is certain that Ann is event-rational. And so on. This yields our RCBER ((Event) Rationality and Common Belief of Event Rationality) set of states.

Event-rationality captures the idea of choosing a strategy with extra validation, in the sense that a strategy has to be optimal under one's conjecture, but also pass a series of validating tie-breaking tests. We also introduce the idea of extra validation of a belief. Consider a type t^a that believes that an event $E \in S^b \times T^b$ is true, and is associated with a list ℓ of subsets of S^b. The belief on the event E will be validated by the list ℓ if there is an element of the list, say $E^b \in \ell$, that is equal to the projection of E on S^b.

Event-rationality and common validated belief of event-rationality is again captured as the intersection of infinitely many events: Ann and Bob are event-rational. Ann has a validated belief that Bob is event-rational and Bob has a validated belief that Ann is event-rational. And so on. This yields our RCvBER ((Event) Rationality and Common validated Belief of Event Rationality) set of states.

Our results are as follows. We characterize the strategies that are compatible with RCBER by a solution concept, hypo-admissible sets (HAS), which is related to the self-admissible sets (SAS) of BFK but it is neither weaker or stronger. In a complete structure, RCBER produces the set of strategies that survive one round of elimination of inadmissible strategies followed by iterated elimination of strongly dominated strategies ($S^\infty W$). We characterize RCvBER with a solution concept we call hypo-iteratively admissible sets (HIA). In a complete type structure, the resulting set of strategies is precisely the set of iterated admissible strategies (IA). We then show that strategies played under RCvBER constitute an SAS, but the converse is not necessarily true. Because BFK have shown that every SAS is the implication of RCAR in some type structure, the RCvBER construction is more restrictive than the RCAR construction of BFK. Nevertheless, we show that the RCBER and the RCvBER are nonempty whenever the type structure is complete, continuous and compact, therefore providing epistemic criteria for $S^{\infty}W$ and IA.

Our approach provides an alternative and effective perspective to deal with common "knowledge" of admissibility in games. A solution to the inclusion-exclusion problem is obtained by using event-rationality together with having S^b (from Ann's perspective) as one of the tie-breaking sets. LPS-based approaches also obtain a solution to the inclusion-exclusion problem. But some conclusions coming from the LPS-based approach are functions of the notions of rationality and beliefs adopted by the approach. For instance, from BFK and Keisler and Lee (2011) we get that either continuity or completeness have to be dropped for an epistemic characterization of IA to be obtained. Our results show that, using a different notion of rationality, neither continuity nor completeness have to be dropped for such a characterization to be obtained. We should also note that completeness captures the idea that players have no prior knowledge about each other, so it is a desirable property in an epistemic analysis. And continuity is a consequence of the (universal) construction of beliefs about beliefs (c.f. Mertens and Zamir (1985)).

1.1 Related Literature

Bernheim (1984) and Pearce (1984) argue that common knowledge of rationality implies (in terms of behavior) the iteratively undominated (IU) set, that is, the set of strategy profiles surviving iterated deletion of strongly dominated strategies. Tan and Werlang (1988) provides epistemic conditions for UI by characterizing RCBR (rationality and common belief of rationality). Admissibility, or the avoidance of weakly dominated strategies, has a long history in decision and game theory (see Wald (1939), Luce and Raiffa (1957) and Kohlberg and Mertens (1986)). However, Samuelson (1992) shows that common knowledge of admissibility is not equivalent to iterated admissibility and does not always exist. Foundations for the $S^{\infty}W$ strategies (Dekel and Fudenberg (1990)) are provided by Börgers (1994) (using approximate common knowledge), Brandenburger (1992) (using lexicographic probability systems (Blume et al. (1991)) and 0-level belief) and Ben-Porath (1997) (in extensive form games). Stahl (1995) defines the notion of lexicographic rationalizability and shows that it is equivalent to iterated admissibility.

BFK use lexicographic probability systems and characterize rationality and common assumption of rationality (RCAR) by the solution concept of self-admissible sets. They show that rationality and m-th order assumption of rationality is characterized by the strategies that survive $m + 1$ rounds of elimination of inadmissible strategies, in complete type structures.[2] Finally, RCAR is empty in a complete and continuous lexicographic type structure when the agent is not indifferent. Hence, although the IA set can be captured by RmAR (rationality and m-th order assumption of rationality) for big enough m (note that games are finite), BFK do not provide an epistemic criterion for IA. Keisler and Lee (2011) show that the emptiness of RCAR can be overcome if one drops continuity. Yang (2009) provides an epistemic criterion for IA, with an analogous version of BFK's RCAR, that makes use of a weaker notion of "assumption". The message from Keisler and Lee (2011) and Yang (2009), is that continuity strengthens the notion of caution implied by fully supported LPS. The notion of caution implied by event-rationality is independent of continuity.

The paper is organized as follows. In the following section we illustrate the differences between the various notions of rationality and belief through examples. In Sections 3 and 4 we set up the framework and provide the relevant definitions, including event-rationality, RCBER and RCvBER. In Section 5 we characterize RCBER and show that RmBER (m rounds of mutual belief) generates $S^\infty W$, for big enough m. In Section 6 we characterize RCvBER, show that it is more restrictive than RCAR of BFK and show that RmvBER generates the IA set, for big enough m. In Section 7 we show that RCBER and RCvBER are always nonempty in compact, complete and continuous type structures, therefore providing epistemic criteria for $S^\infty W$ and IA. Finally, the Appendix provides decision theoretic foundations for event-rationality and validated beliefs.

2 Examples

In order to illustrate the differences between the BFK approach and that of the present paper, consider the following game from Samuelson (1992) and BFK. There are two players, Ann and Bob.

[2]See Section 6.1 for the formal definition of "assumption".

		1	[1]
		L	R
1	U	$1, 1$	$0, 1$
[1]	D	$0, 2$	$1, 0$

From the literature we know that rationality and common belief of rationality (RCBR) is characterized by the best response sets (BRS) and, in a complete structure, the strategies that survive iterated deletion of strongly dominated strategies.[3] Can we get a similar result for the admissible strategies and the iteratively admissible strategies if we modify the notions of belief and of rationality? Recall that a strategy is admissible if and only if it is a best response to a full support measure (no action of the other player is excluded). Then, the obvious solution is to specify that rationality incorporates full support beliefs.

But such a specification does not always work. In the game above, if Ann is rational, she assigns positive probability to Bob playing L and R. If Bob is rational, he assigns positive probability to Ann playing U and D. Hence, Bob plays L. If Ann knows that Bob is rational, she assigns positive probability only on Bob playing L. But then, Ann is not rational! In other words, the modified RCBR set is empty for this game.

One solution is obtained using lexicographic beliefs. Suppose Ann's primary hypothesis assigns probability 1 to Bob playing L, and her secondary hypothesis assigns probability 1 to Bob playing R. Bob's primary hypothesis assigns 1 on U and his secondary hypothesis assigns 1 on D. Then, Bob playing L is rational because he is indifferent between L and R given his primary measure, but strictly prefers L given his secondary measure.[4] Ann playing U is rational because U is the best response given her primary measure. She assumes that Bob is rational, because she considers Bob playing L infinitely more likely than Bob playing R.[5] Similarly, Bob assumes that Ann is rational. As a result, rationality and common assumption of rationality (RCAR) is nonempty.

A similar result can be obtained if we use the definition of event-rationality in the context of standard type structures. Suppose Ann's belief assigns probability 1 to Bob playing L and Bob's belief μ assigns probability 1 to Ann playing U. Moreover, Bob has the set $S^a \setminus \mathrm{supp}\ \mu$

[3] $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

[4] That is, the associated sequence of payoffs under L is lexicographically greater than the sequence under R.

[5] For more information on the notions of "assumption" and "infinitely more likely", see BFK.

in his tie-breaking list. Bob playing L is event-rational because he plays best response given his beliefs and, although L and R are outcome-equivalent under his support, L is better under a conjecture with support $S^a \setminus \text{supp } \mu$. Similarly, Ann is event-rational since, under her conjecture, she does not need to break ties. Finally, Ann believes that Bob is event-rational and Bob believes that Ann is event-rational. Hence, rationality and common belief of event-rationality (RCBER) is nonempty.

In the game above RCAR and RCBER produce the same strategies because the IA and the $S^\infty W$ sets are equal. However, this is not always true. Consider the following game which illustrates the difference between RCBER (which yields the $S^\infty W$ set) and RCvBER (which yields the IA set).

	L	R
U	$1,0$	$1,3$
M	$0,2$	$2,2$
D	$0,4$	$1,1$

Since D is strongly dominated, event-rational Ann will not play that strategy. In a complete structure though, event-rational Ann will play U or M, while event-rational Bob will play L or R. For example, Ann's type playing U is event-rational if she assigns probability 1 to Bob playing L. Ann's type playing M is also event-rational if she assigns probability 1 to Bob playing R. Note that Ann never needs to break ties. Moreover, for both U and M there are event-rational types of Ann's who assign positive probability to event-rational types of Bob playing L or R. And similarly for Bob. In other words, these types of Ann believe the event "Bob is event-rational", Bob's types believe the event "Ann is event-rational", and so on for any finite order of beliefs about beliefs. Hence, event-rationality and common belief of event-rationality (RCBER) yields the $S^\infty W$ set, $\{\text{U}, \text{M}\} \times \{\text{L}, \text{R}\}$.

Suppose we repeat the same procedure but now impose a stronger form of belief. Take an event $E \subseteq S^b \times T^b$, where S^b, T^b is the set of Bob's strategies and types, respectively. A type t^a of Ann is associated with a belief over $S^b \times T^b$ and a list ℓ of subsets of S^b. We say that t^a has a validated belief in an event E if it assigns probability 1 to E and there exists an element E^b of the list ℓ that is equal to the projection of E on S^b. Imposing event-rationality and common validated belief of event-rationality gives us RCvBER.

Which strategies are generated by RCvBER? The first round of RCvBER yields the set of event-rational types for Ann and event-rational types for Bob, just like RCBER. But the second round of RCvBER requires that each of Ann's types has a validated belief in the event "Bob is rational", and similarly for Bob. Then, all types playing L are excluded. To see this, note that if Bob is event-rational and has a validated belief in the event "Ann is event-rational", then the strategies played by event-rational types of Ann's, namely $\{U, M\}$, must belong to his list. The only event-rational types of Bob playing L (and having a validated belief that Ann is event-rational) are the ones that assign probability 1 on Ann playing M. In order to have a validated belief in $\{U, M\} \times T_0^a$, where T_0^a is Ann's event-rational types, Bob must have U as a tie-breaking set in his list. Moreover, he assigns probability 1 to M and therefore needs to break ties, because L and R are outcome equivalent given his support. But L is never a best response for any conjecture with support on U. Hence, Bob, assigning probability one on M, cannot have a validated belief that Ann is event-rational.

In the third round of RCvBER, Ann has a validated belief that Bob has a validated belief that Ann is event-rational. This means that types of Ann's playing U are excluded, because those types assign positive probability to Bob's types playing L, and none of them has a validated belief that Ann is event-rational. The only event-rational types of Ann playing M and of Bob playing R survive all rounds of RCvBER and generate the IA set, $\{\text{M}\} \times \{\text{R}\}$.

3 Setup

Let (S^a, S^b, π^a, π^b) be a two-player finite strategic form game, with $\pi^a : S^a \times S^b \to \mathbb{R}$, and similarly for b (as usual, a stands for Ann, and b stands for Bob). For any given topological space X, let $\Delta(X)$ denote the space of probability measures defined on the Borel subsets of X, endowed with the weak* topology. We extend π^a to $\Delta(S^a) \times \Delta(S^b)$ in the usual way: $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b)\in S^a\times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$. Similarly for π^b. A strategy $s^a \in S^a$ is a best response to a conjecture $v \in \Delta(S^b)$ if $\pi^a(s^a, v) \geq \pi^a(\hat{s}^a, v)$ for every $\hat{s}^a \in S^a$. It is denoted by $s^a \in BR^a(v)$. Similarly for b.

3.1 Admissibility and Event-Rationality

The following definition and Lemma are taken from BFK.

Definition 1. *Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is* **weakly dominated** *with respect to $X \times Y$ if there exists $\sigma^a \in \Delta(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for* *every $s^b \in Y$ and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is* **admissible** *with respect to $X \times Y$. If s^a is admissible with respect to $S^a \times S^b$, simply say that s^a is admissible.*

Lemma 1. *A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there exists $\sigma^b \in \Delta(S^b)$, with supp $\sigma^b = Y$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

Lexicographic beliefs have been used in dealing with the inclusion-exclusion issue identified by Samuelson (1992) (see BFK, Brandenburger (1992), Stahl (1995), Keisler and Lee (2011) and Yang (2009)). We follow an alternative approach, based on "tie-breaking lists." We stress that our approach is a way of capturing admissible behavior (Lemma 2 below.) Admissible behavior can be viewed as the requirement that ties be broken by events outside the conjecture of a player. This leads us to consider tie-breaking events, as follows.

By a **list of subsets** of S^b we mean a collection $\ell = \{F_1, ..., F_k\}$, with $F_i \subset S^b$ for every $i = 1, ..., k$, for some $k \geq 1$, with the property that $F_i \neq F_j$ for every distinct pair $i, j \in \{1, ..., k\}$. The collection of all such lists, L^b, is a set of finite cardinality, because S^b is a finite set. Similarly for b, with lists ℓ of distinct subsets of S^a denoted by L^a.

For a given conjecture $v \in \Delta(S^b)$, let $\sigma^a \sim_{\text{supp } v} s^a$ denote that the mixed strategy $\sigma^a \in \Delta(S^a)$ satisfies $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for every $s^b \in \text{supp } v$. That is, $\sigma^a \sim_{\text{supp } v} s^a$ means that σ^a is outcome equivalent to s^a in supp v.

Definition 2. *A strategy $s^a \in S^a$ is event-rational if there exists a conjecture $v \in \Delta(S^b)$ and a list $\ell \in L^b$ such that:*

- *$s^a \in BR^a(v)$,*
- *for each $F \in \ell$ with $F \setminus \text{supp } v \neq \emptyset$ and mixed strategy $\sigma^a \in \Delta(S^a)$ with $\sigma^a \sim_{\text{supp } v} s^a$, there exists a conjecture $v' \in \Delta(S^b)$ with supp $v' = F \setminus \text{supp } v$ such that $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$,*
- *$S^b \in \ell$.*

Likewise for b.

The idea is that Ann uses each of the sets in the list ℓ to break ties: whenever she has a conjecture $v \in \Delta(S^b)$ over Bob's choices under which s^a is optimal and s^a is outcome-equivalent to a (mixed) strategy σ^a in supp v, Ann uses each $F \in \ell$ as the "tie-breaking experiments": there has to exist a probability measure v' with support on $F \setminus \text{supp } v$ that validates the choice of s^a. Ann is fully confident in her conjecture v and in her best response s^a to v as long as there is no σ^a that is outcome equivalent to s^a in supp v. In that case, her probabilistic assessments captured by v are irrelevant, for whichever other conjecture $\hat{v}$ with supp $\hat{v} =$ supp v would not help Ann breaking ties between s^a and σ^a. In that case, Ann uses the tie breaking list ℓ.

It is important to note that, although the "tie-breaking experiments" are additional thought experiments that Ann uses to guide her choices, they do not play the role of additional hypotheses in a lexicographic framework. If s^a is indifferent to σ^a according to v, but not outcome equivalent in supp v, then there is no need to break ties. The following Lemma shows the connection between admissibility and event-rationality.

Lemma 2. *For each $F \in \ell$, if s^a is event-rational under ℓ and v such that supp $v \subseteq F$, then s^a is admissible with respect to $S^a \times F$. Conversely, if s^a is admissible with respect to $S^a \times F$, for each $F \in \ell$ and $S^b \in \ell$, then s^a is event-rational under ℓ.*

Proof. Suppose that s^a is event-rational for v such that supp $v \subseteq F$. If supp $v = F$ then the result is immediate so suppose supp $v \subset F$ and $F \setminus \text{supp } v \neq \emptyset$. Suppose there exists $\sigma^a \in \Delta(S^a)$ with $\pi(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in F$, with strict inequality for some $s^b \in F$. Because $s^a \in BR^a(v)$, we have $s^a \sim_{\text{supp } v} \sigma^a$, which implies that there exists v' with supp $v' = F \setminus \text{supp } v$ and $\pi(s^a, v') \geq \pi(\sigma^a, v')$, a contradiction. Conversely, because s^a is admissible with respect to $S^a \times S^b$, there exists v with supp $v = S^b$ such that $s^a \in BR(v)$. Moreover, for each $F \in \ell$ we have $F \setminus \text{supp } v = \emptyset$.

□

Turn now to decision theoretic considerations. We postulate that a decision maker (Ann) may contemplate several theories. She has a theory captured by her preference relation $\succsim$ and the resulting probability measure μ. Let $F_0 =$ supp μ and write $\succsim$ as $\succsim_0$. Moreover, when faced with a comparison between two acts that are completely indifferent according to her theory, Ann resorts to alternate theories, or *gedankenexperiments* (thought experiments). This is captured by a list of conditional preferences, where the conditioning event is outside F_0. Formally, Ann's theories are captured by a list of preferences $(\succsim_0, \succsim_1, \ldots, \succsim_k)$ and the resulting supports $(F_0, ..., F_k)$. F_0 represents *the* theory that guides her choices, while $(F_1, ..., F_k)$ are thought experiments, used only for the purposes of breaking complete indifference. Put differently, F_0 describes Ann's frame of mind, as it contains the states that Ann considers possible, and $(F_1, ..., F_k)$ describe zero probability "counter-factuals" as $F_0 \cap F_i = \emptyset$ for each $i = 1, \ldots, k$. Ann prefers an act x to an act y if $x \succsim_0 y$ and if x is outcome-equivalent to y in F_0, then $x \succsim_i y$ for all $i = 1, ..., k$. Appendix A provides a more detailed exposition and shows that the notion just defined is equivalent to event-rationality.

3.1.1 *Gedankenexperiments*

It is important to stress that each F_i, $i > 0$, is considered impossible by Ann, as it is the support of a preference conditional on an event which is disjoint from her support, F_0. Resorting to an alternative theory to break ties does not entail considering the alternative theory possible. This is obviously true when dealing with "facts": for instance, one may wonder what would have happened if Germany had won World War II, and use it to help deciding whether to move to Germany or not. But one knows that Germany did not win. So the counter-factual "what if Germany had won" is simply a mental construct, and the decision maker is sure that it is impossible. With beliefs, the distinction is not so sharp, because there is no presumption that a conjecture will necessarily come to pass. Still, a decision maker that is fully confident in her conjecture may contemplate alternative scenarios, as a way to validate her planned choices, without considering the alternative scenarios real possibilities. As an extreme example, consider the same decision to move or not to Germany, and say that under the scenario considered by the decision maker, she is completely indifferent between the two options. Now consider the alternative scenario of Martians invading the Earth. And consider that the decision maker is fully confident that this is impossible. Still, say that in her mind living in Germany would be the best way to be protected from Martians. This alone may tip the scale in favor of moving to Germany.

To repeat, our postulate is that resorting to thought experiments does not entail considering the events used in the thought experiment possible. The following example, suggested by an anonymous referee, illustrates this point further.

	L	C	R
U	$4, 6$	$0, 0$	$4, 3$
M	$0, 0$	$4, 6$	$0, 3$
D	$2, 3$	$2, 3$	$0, 0$

Suppose that Ann is represented by a measure with support $F_0 = \{$L,C$\}$ and she resorts to the *gedankenexperiment* $F_1 = \{\text{R}\}$ to resolve complete indifference. Ann's subjective belief assigns 50% probability to L and C respectively. Conditional on F_1, Ann's subjective belief assigns 100% probability to R. D is outcome equivalent to a coin-flip between U and M under $\succsim_0$, so Ann cannot decide between D and this coin-flip, and resorts to F_1 for help. Under $\succsim_1$, D is strongly dominated by the coin flip, so the coin flip is preferred to D (equivalently, Ann's tie-breaking list consists of the set R, and there's no conjecture supported in R that makes D better than the coin flip, so D is not event-rational). Note that R is weakly dominated by a coin flip between L and C. So Ann resorts to a thought experiment whereby Bob plays an inadmissible strategy. But, as we indicated above, this does not mean that Ann does not believe that Bob plays admissible: her theory only considers possible Bob playing either L or C, which are admissible. So Ann believes that Bob plays admissible, and at the same time Ann uses an alternative theory to help break ties.

Moreover, the alternative theories are not restricted to be measurable with respect to Bob's rationality (or lack of it). The same referee suggested the following modification of the example:

	L	C	R	E
U	$4,6$	$0,0$	$4,3$	$4,6$
M	$0,0$	$4,6$	$0,3$	$0,0$
D	$2,3$	$2,3$	$0,0$	$0,3$

Imagine again that Ann is represented by two preferences, with $F_0 = \{\text{L,C}\}$ and $F_1 = \{\text{R,E}\}$ (the second being the *gedankenexperiment* used to break ties). Conditional on F_1 Ann's subjective belief assigns 50% probability to R and E respectively, and $\succsim_0$ is as above. Ann again decides for the coin flip between U and M over D by resorting to $\succsim_1$, which is a thought experiment that envisages Bob playing an admissible strategy E and an inadmissible strategy R. Yet again, Ann knows that Bob plays admissible (either L or C).

What is at stake here is our perspective over thought experiments. Instead of having "infinitely less likely events" represent what Ann believes is impossible and yet possible, we fix that Ann only considers F_0 possible. The *gedankenexperiments* are the events $\{F_1, ..., F_k\}$, which Ann believes are impossible. Yet, Ann uses the information coming from these thought experiments to help break ties.

Note that because Ann is not indifferent between two strategies that are outcome equivalent under her support, she "considers everything to be possible" in terms of how she acts. However, when reasoning about Bob, she uses her measure μ (and possibly an additional tie-breaking set to validate her beliefs) and therefore believes that Bob is event-rational. The combination of considering everything possible and believing that Bob is event-rational resolves the inclusion/exclusion tension. In the two examples above, Ann's theory only considers Bob playing admissible strategies, so Ann includes only admissible strategies. At the same time, event-rational Ann breaks ties with thought experiments that envisage Bob playing either admissible or inadmissible strategies, so Ann does not have to include all of Bob's strategies in her frame of mind.

3.2 Type Structures and Beliefs

Type structures are used to describe interactive beliefs. Because event-rationality has players using tie-breaking sets, a type of a player must determine a conjecture and a list of tie-breaking sets. Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$.

Definition 3. *An (S^a, S^b)-based type structure with tie-breaking lists is a structure*

$$\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle,$$

where $\lambda^a : T^a \to \Delta(S^b \times T^b) \times L^b$, and similarly for b. Members of T^a, T^b are called types, members of L^a, L^b are called lists and members of $S^a \times T^a \times S^b \times T^b$ are called states.

We refer to an (S^a, S^b)-based type structure with tie-breaking lists as simply a type structure. The types spaces T^a and T^b are assumed topological. The sets S^a, S^b, L^a, L^b are finite, and we endow each with the discrete topology so that they are compact spaces. The belief mappings λ^a and λ^b are assumed Borel measurable. A type structure is: **complete** when λ^a and λ^b are surjective (c.f. Brandenburger (2003)); **continuous** when λ^a and λ^b are continuous; and **compact** when T^a and T^b are compact spaces.

The by now standard construction of all coherent hierarchies of "beliefs about beliefs" yields a complete, continuous and compact type structure. So existence of such structures (which we assume in some of our results below) is guaranteed. Some details are provided in Appendix B.

We use the notation $\lambda^a(t^a) = (\mu^a(t^a), \ell^a(t^a))$, with $\mu^a(t^a) \in \Delta(S^b \times T^b)$ and $\ell^a(t^a) \in L^b$. Similarly for b.

Fix an event $E \subseteq S^b \times T^b$ and write

$$B^a(E) = \{t^a \in T^a : \mu^a(t^a)(E) = 1\}$$

as the set of types that are certain of the event E. This is the standard definition of certainty (as 1-belief): the states of Bob are the strategy type pairs in $S^b \times T^b$, and Ann's beliefs are over Bob's states. Note that B^a satisfies monotonicity: if Ann is certain of E and $E \subset F$ then Ann is also certain of F.

Fix $E \subseteq S^b \times T^b$ and define the following operator

$$B^a_*(E) = \{t^a \in T^a : \mathrm{proj}_{S^b} E \in \ell^a(t^a)\}.$$

We say that a type of Ann's has a **validated belief** in an event $E \subseteq S^b \times T^b$ if the type belongs to the set

$$B^a_v(E) = B^a(E) \cap B^a_*(E).$$

Appendix A provides a preference based characterization of validated beliefs.

3.2.1 Lists made of Subsets of Strategies Suffice for Breaking Ties

Before moving on, let us stress the following important property. The principle behind event-rationality is that a player goes beyond her "frame of mind" to break ties. With a formal type structure, the frame of mind is given by a type t^a and its associated assessment $\mu^a(t^a)$ over $S^b \times T^b$ (note that the list $\ell^a(t^a)$ captures what's is beyond the frame of mind). So one could argue that we should consider lists over subsets of $S^b \times T^b$, thereby treating strategies and types symmetrically. In fact, the inclusion/exclusion tension identified by Samuelson (1992) could be interpreted as requiring that the player includes "everything else" in her thought experiments.[6]

But it is redundant to include lists of subsets of $S^b \times T^b$ for tie-breaking purposes: a list ℓ made of subsets E^b of S^b breaks ties between s^a and σ^a if, and only if, a list $\hat{\ell}$ made of subsets E of $S^b \times T^b$ whose projections on S^b are given by the subsets E^b of the list ℓ also breaks ties between s^a and σ^a. This is obvious: types are payoff irrelevant.

[6]This logic is employed in BFK.

Moreover, if one insists in using lists $\hat{\ell}$ of subsets of $S^b \times T^b$, the analysis below would follow on exactly the same lines, defining validated beliefs using the operator

$$\hat{B}^a_*(E) = \{t^a \in T^a : E \in \hat{\ell}^a(t^a)\}$$

in the place of the operator B^a_*, where $\hat{\ell}^a(t^a)$ would denote the list of subsets of $S^b \times T^b$ associated with type t^a. In fact, as we just argued, tie-breaking purposes would not restrict the "type" component of the lists $\hat{\ell}$. In Appendix B we show that nothing relevant would be changed in the analysis below. Thus, the seemingly asymmetric treatment of strategies and types is irrelevant, as a symmetric analysis can be provided with the appropriate changes in notation.

3.3 RCBER - Rationality and Common Belief of Event-Rationality

With type structures, a state for Ann is a pair (s^a, t^a) determining what she plays (s^a) and her state of mind (t^a). We extend the definition of event-rationality to strategy-type pairs as follows:

Definition 4. *Strategy-type pair $(s^a, t^a) \in S^a \times T^a$ is event-rational if*

- $s^a \in BR^a(v)$, *for* $v = marg_{S^b}\mu^a(t^a)$,
- *for each* $F \in \ell^a(t^a)$ *with* $F \setminus supp\ v \neq \emptyset$ *and mixed strategy* $\sigma^a \in \Delta(S^a)$ *with* $\sigma^a \sim_{supp\ v} s^a$, *there exists a conjecture* $v' \in \Delta(S^b)$ *with* $supp\ v' = F \setminus supp\ v$ *such that* $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$,
- $S^b \in \ell^a(t^a)$.

Likewise for b.

Let R^a_1 be the set of event-rational strategy-type pairs (s^a, t^a). For finite m, define R^a_m inductively by

$$R^a_{m+1} = R^a_m \cap [S^a \times B^a(R^b_m)].$$

Similarly for b.

Definition 5. *If $(s^a, t^a, s^b, t^b) \in R^a_{m+1} \times R^b_{m+1}$, say there is event-rationality and mth-order belief of event-rationality (RmBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} R^a_m \times \bigcap_{m=1}^{\infty} R^b_m$ say there is event-rationality and common belief of event-rationality (RCBER) at this state.*

In words, there is RCBER at a state if Ann is event-rational, Ann believes that Bob is event-rational, Ann believes that Bob believes that Ann is event-rational, and so on. Similarly for Bob. Believing that Bob is event-rational means that Ann is certain that Bob only chooses strategies that are best responses to Bob's conjectures that Ann considers possible, and that Bob breaks ties using the sets of strategies in his list.

Note that for a strategy-type pair (s^a, t^a) to belong to R^a_m the following conditions are satisfied. Strategy s^a is a best response to $v = \text{marg}_{S^b}\mu^a(t^a)$, $\mu^a(t^a)(R^b_{m-1}) = 1$ and whenever $\sigma^a \sim_{\text{supp } v} s^a$, for each $E^b \in \ell^a(t^a)$, there exists a conjecture v' in $E^b \setminus \text{supp } v$ for which $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. Notice that Ann is certain that the conjectures of Bob are of the form $v = \text{marg}_{S^a}\mu^b(t^b)$, for $t^b \in \text{proj}_{T^b} R^b_{m-1}$, and knows that, for each such conjecture, Bob breaks each tie using some v' in $E^b \setminus \text{supp } v$. We show below that this flexibility implies that the set of strategies compatible with RCBER are the ones that survive one round of elimination of inadmissible strategies, followed by iterated elimination of strongly dominated strategies.

3.4 RCvBER - Rationality and Common validated Belief of Event-Rationality

Let $\overline{R}^a_1$ be the set of event-rational strategy-type pairs (s^a, t^a). For finite m, define $\overline{R}^a_m$ inductively by

$$\overline{R}^a_{m+1} = \overline{R}^a_m \cap [S^a \times B^a_v(\overline{R}^b_m)].$$

Similarly for b.

The only difference with RCBER is that we use the validated belief operator instead of the standard one.

Definition 6. *If $(s^a, t^a, s^b, t^b) \in \overline{R}^a_{m+1} \times \overline{R}^b_{m+1}$, say there is event-rationality and mth-order consistent belief of event-rationality (RmvBER) at this state. If $(s^a, t^a, s^b, t^b) \in \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \bigcap_{m=1}^{\infty} \overline{R}^b_m$ say there is event-rationality and common consistent belief of event-rationality (RCvBER) at this state.*

Because validated beliefs are stronger than standard beliefs, RCvBER $\subseteq$ RCBER.

Note again that RCBER and RCvBER avoid the inclusion-exclusion tension. What a type t^a of Ann believes about Bob's choices is given by the marginal of $\mu^a(t^a)$ over S^b. And a type that knows that Bob's strategy-type pairs are in $\overline{R}^b_m$ is a type that assigns positive probability only to the strategies that are consistent with $\overline{R}^b_m$. So many of Bob's strategies can be excluded from t^a's consideration, without causing any contradiction in the construction. The event-rational (s^a, t^a) resorts to the tie-breaking list $\ell^a(t^a)$ to handle counter-factuals, without having to believe that the counter-factuals are a real possibility.

4 Solution Concepts

4.1 Self-Admissible and Hypo-Admissible Sets

By construction, event-rationality implies playing admissible strategies. If we add common belief of event-rationality, then the solution concept is that of a hypo-admissible set (HAS) that we define below. We compare the HAS with several solution concepts that have been proposed in the literature. But first a definition.

Definition 7. *Say that r^a supports s^a given Q^b if there exists some $\sigma^a \in \Delta(S^a)$ with $r^a \in$ supp σ^a and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in Q^b$. Write $su_{Q^b}(s^a)$ for the set of $r^a \in S^a$ that supports s^a given Q^b. Likewise for b.*

This is a generalization of the definition in BFK of the support of a strategy s^a, which they denote $\mathrm{su}(s^a)$. In particular, $\mathrm{su}_{S^b}(s^a) = \mathrm{su}(s^a)$.

BFK characterize rationality and common assumption of rationality (RCAR) by the solution concept of a self-admissible set (SAS).

Definition 8. *The set $Q^a \times Q^b \subseteq S^a \times S^b$ is an SAS if:*

- *each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$,*
- *each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$,*
- *for any $s^a \in Q^a$, if $r^a \in su_{S^b}(s^a)$, then $r^a \in Q^a$.*

Likewise for b.

In particular, BFK show that the projection of the RCAR into $S^a \times S^b$ is an SAS. Conversely, given an SAS $Q^a \times Q^b$, there is a type structure such that the projection of RCAR into $S^a \times S^b$ is equal to $Q^a \times Q^b$. BFK discuss the need for the third requirement in the definition of an SAS. In particular, consider the weak best response sets (WBRS), which does not include a restriction on convex combinations.

Definition 9. *The set* $Q^a \times Q^b \subseteq S^a \times S^b$ *is a WBRS if:*

- *each* $s^a \in Q^a$ *is admissible with respect to* $S^a \times S^b$*,*
- *each* $s^a \in Q^a$ *is not strongly dominated with respect to* $S^a \times Q^b$*.*

Likewise for b.

As Brandenburger (1992) and Börgers (1994) show, if common assumption of rationality is relaxed to common belief at level 0 of rationality (RCB0R) (that is, believing E means $\mu_0(E) = 1$, where μ_0 is the first measure of the agent's LPS), then the projection of RCB0R into $S^a \times S^b$ is a WBRS. Conversely, given a WBRS $Q^a \times Q^b$, there is a type structure such that $Q^a \times Q^b$ is contained in (but not necessarily equal to) the projection of RCB0R into $S^a \times S^b$.[7]

We are now ready to introduce the solution concept of hypo-admissible sets (HAS).

Definition 10. *The set* $Q^a \times Q^b \subseteq S^a \times S^b$ *is an HAS if:*

- *each* $s^a \in Q^a$ *is admissible with respect to* $S^a \times S^b$*.*

  For each $s^a \in Q^a$ *there is nonempty* $Q_0 \subseteq Q^b$ *such that*

- s^a *is admissible with respect to* $S^a \times Q_0$*,*
- *for any* $s^a \in Q^a$*, if* $r^a \in su_{Q_0}(s^a)$ *and* r^a *is admissible with respect to* $S^a \times S^b$ *then* $r^a \in Q^a$*.*

Likewise for b.

Note that the first two properties for a WBRS are equivalent to the first two properties for an HAS and they are implied by the first two properties for an SAS. Hence, the SAS and the HAS are always WBRS but the opposite does not hold. Moreover, an SAS is not necessarily an HAS and an HAS is not necessarily an SAS. The differences between the HAS and the SAS can be further illustrated by the following two solution concepts. The first is $S^\infty W$, the set of strategies that survive one round of deletion of inadmissible strategies followed by iterated deletion of strongly dominated strategies (Dekel and Fudenberg (1990)).

[7] See Section 11 in BFK.

Definition 11. *Let $SW_1^i = S_1^i$, for $i = a, b$ be the set admissible strategies and define inductively for $m \geq 1$,*

$$SW_{m+1}^i = \{s^i \in SW_m^i : s^i \text{ is not strongly dominated with respect to } SW_m^a \times SW_m^b\}.$$

Let $S^\infty W = \bigcap_{m=1}^\infty SW_m^a \times \bigcap_{m=1}^\infty SW_m^a$.

The second is the set of strategies that survive iterated deletion of weakly dominated strategies, the IA set.

Definition 12. *Set $S_0^i = S^i$ for $i = a, b$ and define inductively*

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is admissible with respect to } S_m^a \times S_m^b\}.$$

A strategy $s^i \in S_m^i$ is called m-admissible. A strategy $s^i \in \bigcap_{m=0}^\infty S_m^i$ is called iteratively admissible (IA).

We then have that the $S^\infty W$ set is both an HAS and a WBRS (but not an SAS) and the IA set is an SAS and a WBRS (but not a HAS). The following game from Section 2 illustrates the various definitions.

	L	R
U	$1, 0$	$1, 3$
M	$0, 2$	$2, 2$
D	$0, 4$	$1, 1$

The IA set is $\{M\} \times \{R\}$. It is an SAS but not an HAS, because although $L \in \text{su}_{\{M\}}(R)$ and L is admissible, it does not belong to the IA set. Moreover, $S^\infty W = \{U, M\} \times \{L, R\}$ is an HAS but not an SAS, because L is not admissible with respect to $\{U, M\}$. That is, in a sense the SAS captures IA whereas the HAS captures $S^\infty W$.

4.2 Generalized Self-Admissible and Hypo-Iteratively Admissible Sets

In Section 5 we show that HAS characterizes RCBER with $E = S$. With a view to obtain a characterization of RCvBER and to relate it to the concepts presented above, we introduce the following two solution concepts.

Definition 13. *The set* $Q^a \times Q^b \subseteq S^a \times S^b$ *is an* $SAS_{P^a \times P^b}$ *if:*

- *each* $s^a \in Q^a$ *is admissible with respect to* $S^a \times S^b$*,*
- *each* $s^a \in Q^a$ *is admissible with respect to* $S^a \times Q^b$*,*
- *for any* $s^a \in Q^a$*, if* $r^a \in su_{P^b}(s^a)$ *and* r^a *is admissible with respect to* $S^a \times S^b$*, then* $r^a \in Q^a$*.*

Likewise for b.

This is a generalization of the SAS, since the only difference is that the support $\mathrm{su}_{P^b}(s^a)$ is with respect to an abstract set P^b, not S^b. This means that the SAS is equivalent to the $\mathrm{SAS}_{S^a \times S^b}$.[8] Moreover, if $Q^a \times Q^b \subseteq P^a \times P^b$ then an $\mathrm{SAS}_{Q^a \times Q^b}$ is also an $\mathrm{SAS}_{P^a \times P^b}$, but the reverse may not hold. This means that for any $P^a \times P^b$, an $\mathrm{SAS}_{P^a \times P^b}$ is also an SAS. Moreover, an $\mathrm{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is also an HAS.

Definition 14. *A set* $Q^a \times Q^b$ *is a hypo-iteratively admissible (HIA) set if there exist sequences of sets* $\{W_i^a\}_{i=0}^{\infty}$*,* $\{W_i^b\}_{i=0}^{\infty}$*, with* $W_0^a = S^a$*,* $W_0^b = S^b$*, such that for each* $m \geq 0$*,*

- *each* $s^a \in W_{m+1}^a$ *is admissible with respect to* $S^a \times W_m^b$ *and belongs to* W_m^a*,*
- *for any* k*,* m*, where* $k \geq m$*, if* $s^a \in W_{k+1}^a$*,* $r^a \in su_{W_k^b}(s^a) \cap W_m^a$ *and* r^a *is admissible with respect to* $S^a \times W_m^b$*, then* $r^a \in W_{m+1}^a$*,*
- *there is* k *such that for all* $m \geq k$*,* $W_m^a = Q^a$*.*

Likewise for b.

The HIA sets resemble the IA set, with the only difference that one starts with a subset of admissible strategies and always includes the strategies that are equivalent (in the sense of su_Q) to strategies that survive subsequent rounds. Moreover, the HIA can be thought of as an analogue of the best response set (BRS).[9] If we replace admissible with strongly undominated in the definition of HIA then we get a BRS. Conversely, each BRS $Q^a \times Q^b$ can be written as a modified HIA (just set $W_i^a = Q^a$ and $W_i^b = Q^b$ for all $i \geq 1$).

[8] Note that if $r^a \in \mathrm{su}_{S^b}(s^a)$ and s^a is admissible, then r^a is also admissible. Hence, the third condition for a $\mathrm{SAS}_{S^a \times S^b}$ is identical to the third condition for a SAS.

[9] Recall that $Q^a \times Q^b$ is a BRS if each $s^a \in Q^a$ is strongly undominated with respect to $S^a \times Q^b$ and likewise for b.

5 Characterization of RCBER

Propositions 1 and 2 below show that RCBER is characterized by the HAS set in a rich type structure, and that RCmBER generates the $SW^a_m \times SW^b_m$ strategies, for each m, in a complete type structure.

We say that a type structure is **rich** if for each type t^a with $\ell^a(t^a) = (E^b_1, ..., E^b_n)$ and $E^b_1 \supsetneq E^b_2 \supsetneq \ldots \supsetneq E^b_n$, there exists a type t^a_0 with $\ell^a(t^a_0) = (E^b_1, ..., E^b_{n-1})$, and $\mu^a(t^a) = \mu^a(t^a_0)$.

Recall our notation: RCBER is given by $\bigcap_{m=1}^{\infty} R^a_m \times \bigcap_{m=1}^{\infty} R^b_m$.

Proposition 1. *(i) Fix a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} R^a_m \times proj_{S^b} \bigcap_{m=1}^{\infty} R^b_m$ is an HAS.*

(ii) Fix an HAS $Q^a \times Q^b$. Then there is a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b\rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} R^a_m \times proj_{S^b} \bigcap_{m=1}^{\infty} R^b_m$.

Proof. Throughout we keep the convention that for any two sets, E and F, $E \times F = \emptyset$ implies $E = \emptyset$ and $F = \emptyset$. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R^a_m \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R^b_m$ is empty, then the conditions for HAS are satisfied, so suppose that it is nonempty and fix $s^a \in Q^a = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R^a_m$. Then, for some t^a, (s^a, t^a) is consistent with RCBER and s^a is admissible, by Lemma 2. Since t^a believes each R^b_m, for all m, it also believes $\bigcap_{m=1}^{\infty} R^b_m$. From the conjuction and marginalization properties of belief there is $v = \text{marg}_{S^b}\mu^a(t^a)$, with support contained in $\text{proj}_{S^b} \bigcap_{m=1}^{\infty} R^b_m$, such that s^a is optimal under v.

Let $Q_0 = \text{supp } v$. We have that s^a is admissible with respect to $Q_0 = \text{supp } v$, which is a subset of $Q^b = \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R^b_m$. Suppose $s^a \in Q^a$, $r^a \in \text{su}_{\text{supp } v}(s^a)$ and r^a is admissible. From Lemma D.2 in BFK, r^a is optimal under v whenever $(s^a, t^a) \in R^a_1$.[10] Because the type structure is rich, there exists type t^a_0 with $\mu^a(t^a_0) = \mu^a(t^a)$ and $\ell^a(t^a_0) = S^b$. Since r^a is admissible, we have that $(r^a, t^a_0) \in R^a_1$. The same is true for all R^a_m, hence the third property for an HAS is satisfied.

For part (ii) fix an HAS $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to $Q_{s^a} \subseteq Q^b$, there is a v with $\text{supp } v = Q_{s^a}$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \text{su}_{Q_{s^a}}(s^a)$ (Lemma D.4 in BFK).[11] Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that

[10] Lemma D.2 specifies that if F is a face of a polytope P and $x \in F$, then $\text{su}(x) \subseteq F$, where $\text{su}(x)$ is the set of points that support x. The geometry of polytopes is presented in Appendix D in BFK.

[11] Lemma D.4 specifies that if x belongs to a strictly positive face of a polytope P, then $\text{su}(x)$ is a strictly positive face of P.

supp $\mu^a(s^a) = \{(s^b, s^b) : s^b \in Q_{s^a}\}$, $\ell^a(s^a) = \{S^b\}$ and $v = \text{marg}_{S^b}\mu^a(s^a)$ for the v found above. Similarly for b. Note that the type structure is rich.

First, we show that for each $s^a \in Q^a$, (s^a, s^a) is event-rational. By construction, s^a is optimal under $v = \text{marg}_{S^b}\mu^a(s^a)$ and admissible. Hence, (s^a, s^a) is event-rational and $Q^a \subseteq \text{proj}_{S^a} R_1^a$. Suppose $(r^a, t^a) \in R_1^a$, where $t^a = s^a$. Then, $r^a \in \text{su}_{Q_{s^a}}(s^a)$ and r^a is admissible with respect to Q_{s^a}. From Lemma 2, r^a is admissible. From the definition of an HAS this implies that $r^a \in Q^a$ and $Q^a = \text{proj}_{S^a} R_1^a$. Applying similar arguments we have that $Q^b = \text{proj}_{S^b} R_1^b$.

By construction, each $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b) which consists of event-rational strategy-type pairs, hence t^a believes R_1^b and $(s^a, s^a) \in R_2^a$. This implies that $R_2^a = R_1^a$ and likewise for b. Thus, $R_m^a = R_1^a$ and $R_m^b = R_1^b$ for all m, by induction. Since $\text{proj}_{S^a} R_1^a \times \text{proj}_{S^b} R_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} R_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} R_m^b$. $\square$

Proposition 2. *Fix a complete structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m,*

$$\textit{proj}_{S^a} R_m^a \times \textit{proj}_{S^b} R_m^b = SW_m^a \times SW_m^b.$$

Proof. Let T_0^a be the set of types t^a such that $\ell^a(t^a) = \{S^b\}$. From Lemma 2 we have that $(s^a, t^a) \in R_1^a$ implies s^a is admissible. Conversely, since we have a complete structure, if s^a is admissible then there exists $t^a \in T_0^a$ such that $(s^a, t^a) \in R_1^a$. Hence, $\text{proj}_{S^a} R_1^a = S_1^a = SW_1^a$ and $\text{proj}_{S^b} R_1^b = S_1^b = SW_1^b$. Suppose that for up to m we have that $\text{proj}_{S^a} R_m^a = SW_m^a$ and $\text{proj}_{S^b} R_m^b = SW_m^b$. Suppose $s^a \in SW_{m+1}^a$. Then, $s^a \in SW_m^a = \text{proj}_{S^a} R_m^a$. Because s^a is not strongly dominated with respect to $SW_m^a \times SW_m^b$, it is also not strongly dominated with respect to $S^a \times SW_m^b$. Hence, there is a v with supp $v \subseteq SW_m^b$ under which s^a is optimal. We take (s^a, t^a), $t^a \in T_0^a$, with supp $\mu^a(t^a) \subseteq R_m^b$ and $\text{marg}_{S^b}\mu^a(t^a) = v$. Because s^a is admissible with respect to S^b, (s^a, t^a) is event-rational. Because $t^a \in B^a(R_m^b)$ and $R_m^b \subseteq R_k^b$, $1 \leq k \leq m$, we have that $(s^a, t^a) \in R_{m+1}^a$ and $s^a \in \text{proj}_{S^a} R_{m+1}^a$.

Suppose $s^a \in \text{proj}_{S^a} R_{m+1}^a$. Then, $s^a \in SW_m^a = \text{proj}_{S^a} R_m^a$ and supp $\text{marg}_{S^b}\mu^a(t^a) \subseteq SW_m^b = \text{proj}_{S^b} R_m^b$. Because s^a is optimal under v, where supp $v \subseteq SW_m^b$, s^a is not strongly dominated with respect to SW_m^b and therefore $s^a \in SW_{m+1}^a$. $\square$

6 Characterization of RCvBER

Propositions 3 and 4 below show that RCvBER is characterized by the HIA set and RmvBER generates the IA set in a complete type structure, for big enough m.

Recall our notation: RCvBER is given by $\bigcap_{m=1}^{\infty} \overline{R}_m^a \times \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proposition 3.

(i) Fix a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is an HIA set.

(ii) Fix an HIA set $Q^a \times Q^b$. Then there is a rich type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = proj_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times proj_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$.

Proof. For part (i), if $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is empty, then the conditions for an HIA set are satisfied, so suppose that it is nonempty.

Set $W_m^a = \text{proj}_{S^a} \overline{R}_m^a$ for $m \geq 1$ and likewise for b. From Lemma 2, all strategies in $\text{proj}_{S^b} \overline{R}_{m+1}^a$ are admissible with respect to $S^a \times W_m^b$ and, by construction, belong to $\text{proj}_{S^b} \overline{R}_m^a$.

Suppose that for some k, m, where $k \geq m$, we have that $s^a \in W_{k+1}^a = \text{proj}_{S^b} \overline{R}_{k+1}^a$, $r^a \in \text{su}_{W_k^b}(s^a) \cap W_m^a$ and r^a is admissible with respect to $S^a \times W_m^b$. This implies that for some t^a, $(s^a, t^a) \in \overline{R}_{k+1}^a$, where supp $\text{marg}_{S^b} \mu^a(t^a) \subseteq W_k^b$ and list $\ell^a(t^a)$ contains at least all sets W_p^b, for $p \leq m$. Because the type structure is rich, there exists type t_0^a, with $\ell^a(t_0^a)$ that contains all sets W_p^b, for $p \leq m$, and nothing else. Moreover, t_0^a is identical to t^a in all other respects. Since $r^a \in \text{su}_{W_k^b}(s^a)$, r^a is optimal given $\text{marg}_{S^b} \mu^a(t_0^a)$. Moreover, r^a is admissible with respect to $S^a \times W_p^b$, for $p \leq m$.

All these imply that $(r^a, t_0^a) \in \overline{R}_{m+1}^a$. The third condition is satisfied because $\text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is nonempty and the strategies are finite.

For part (ii), fix an HIA set $Q^a \times Q^b$, with sequences of sets $\{W_m^a\}_{m=0}^{m=n'}$, $\{W_m^b\}_{m=0}^{m=n}$, where $W_{n'}^a = Q^a$ and $W_n^b = Q^b$. Construct the following type structure. For each $m \geq 1$, for each $s^a \in W_m^a$, find the measure $v(s^a, m)$ with support on W_{m-1}^b such that r^a is a best response to $v(s^a, m)$ if and only if $r^a \in \text{su}_{W_{m-1}^b}(s^a)$. This is possible because of Lemma D.4 in BFK. Type $t^a(s^a, m)$ has a marginal $v(s^a, m)$ on S^b, the list $\ell^a(t^a(s^a, m)) = \{W_0^b, \ldots, W_{m-1}^b\}$ on L^b (omitting W_{m-j}^b if it is equal to W_{m-j-1}^b) and assigns positive probability only to strategy-types $(s^b, t^b(s^b, m-1))$, for $s^b \in W_{m-1}^b$. Finally, assign to each $s^a \in S^a$ type $t^a(r^a, 0)$ which is equal to $t^a(r^a, k)$, for some $r^a \in W_k^a$, $k > 0$. Similarly for b.

We now show that RCvBER generates the HIA set. For $m = 1$, we show that $\text{proj}_{S^a}\overline{R}_1^a = W_1^a$. Suppose that $s^a \in W_1^a$. Because s^a is admissible and a best response to $v(s^a, 1)$, we have $(s^a, t^a(s^a, 1)) \in \overline{R}_1^a$ and $s^a \in \text{proj}_{S^a}\overline{R}_1^a$. Suppose $r^a \in \text{proj}_{S^a}\overline{R}_1^a$. Then, r^a is a best response to some measure $v(s^a, k+1)$, $k \geq 0$, for $s^a \in W_{k+1}^a$ and $r^a \in \text{su}_{W_k^b}(s^a) \cap W_0^a$. Because $(r^a, t^a(s^a, k+1))$ is event-rational, r^a is admissible. Therefore, by the second property for an HIA set, $r^a \in W_1^a$. Moreover, by construction, for each $s^a \in W_1^a$, $(s^a, t^a(s^a, 1)) \in \overline{R}_1^a$, and similarly for b.

Assume that for up to m, $\text{proj}_{S^a}\overline{R}_m^a = W_m^a$ and for each $s^a \in W_m^a$, $(s^a, t^a(s^a, m)) \in \overline{R}_m^a$. Similarly for b. Suppose that $s^a \in W_{m+1}^a$. By construction, s^a is a best response to $v(s^a, m+1)$, which has a support of $W_m^b = \text{proj}_{S^b}\overline{R}_m^b$, and it is admissible with respect to $S^a \times W_m^b$. Moreover, $\ell^a(t^a(s^a, m+1)) = \{W_0^b, \ldots, W_m^b\}$ and type $t^a(s^a, m+1)$ assigns positive probability only to types $(s^b, t^b(s^b, m)) \in \overline{R}_m^b$, for $s^b \in W_m^b$. This implies that $(s^a, t^a(s^a, m+1)) \in \overline{R}_{m+1}^a$ and $s^a \in \text{proj}_{S^a}\overline{R}_{m+1}^a$. Suppose $r^a \in \text{proj}_{S^a}\overline{R}_{m+1}^a$. By construction, the only measures that have support which is a subset of W_m^b are measures that are associated with strategies s^a that belong to W_{k+1}^a, where $k+1 > m$. Hence, $(r^a, t^a(s^a, k+1)) \in \overline{R}_{m+1}^a$ and r^a is a best response to some measure $v(s^a, k+1)$. By construction, $r^a \in \text{su}_{W_k^b}(s^a)$. Moreover, r^a is admissible with respect to $S^a \times W_m^b$. Hence, by the second property for an HIA set we have that $r^a \in W_{m+1}^a$.

$\square$

Proposition 4. *Fix a complete type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then, for each m,*

$$proj_{S^a}\overline{R}_m^a \times proj_{S^b}\overline{R}_m^b = S_m^a \times S_m^b.$$

Proof. For $m = 1$, Lemma 2 and a complete structure imply $\text{proj}_{S^a}\overline{R}_1^a = S_1^a$. Suppose that for up to m we have that $\text{proj}_{S^a}\overline{R}_m^a = S_m^a$ and $\text{proj}_{S^b}\overline{R}_m^b = S_m^b$. Suppose $s^a \in S_{m+1}^a$. Then, $s^a \in S_m^a = \text{proj}_{S^a}\overline{R}_m^a$. Because s^a is admissible with respect to $S_m^a \times S_m^b$, it is also admissible with respect to $S^a \times S_m^b$. Note that $S_m^b \subseteq \ldots S_1^b \subseteq S^b$ and take t^a such that $\text{marg}_{S^b}\mu^a(t^a) = v$, $\ell^a(t^a) = \{S^b, S_1^b, \ldots, S_m^b\}$. Because s^a is admissible with respect to $S^a \times S_m^b$, we can choose v such that $\text{supp}v = S_m^b$ and s^a is best response to v. Therefore, $\text{supp}\mu^a(t^a) = \overline{R}_m^b$. Take any set $S_i^b \in \ell(t^a)$ with $S_i^b \setminus S_m^b \neq \emptyset$ and mixed strategy σ^a such that $\sigma^a \sim_{S_m^b} s^a$. Suppose there exists no measure v', with $\text{supp}v' = S_i^b \setminus \text{supp}v$, such that $\pi^a(s^a, v') \geq \pi^a(\sigma^a, v')$. Then, σ^a weakly dominates s^a on S_i^b, which implies that s^a is not admissible with respect to $S^a \times S_i^b$, a contradiction. Therefore, (s^a, t^a) is event-rational and $t^a \in B_v^a(\overline{R}_k^b)$ for all $k \leq m$, which implies that $(s^a, t^a) \in \overline{R}^a_{m+1}$ and $s^a \in \text{proj}_{S^a} \overline{R}^a_{m+1}$.

Suppose $s^a \in \text{proj}_{S^a} \overline{R}^a_{m+1}$. Then, $s^a \in S^a_m = \text{proj}_{S^a} \overline{R}^a_m$ and there exists t^a such that $(s^a, t^a) \in \overline{R}^a_{m+1}$ and supp $\text{marg}_{S^b} \mu^a(t^a) \subseteq S^b_m = \text{proj}_{S^b} \overline{R}^b_m$. Because $t^a \in B^a_v(\overline{R}^b_m)$, $S^b_m \in \ell^a(t^a)$. Hence, we have that s^a is admissible with respect to $S^a_m \times S^b_m$ and $s^a \in S^a_{m+1}$. $\square$

6.1 Comparison with BFK

BFK's LPS-based approach uses the following construction. Let $\mathcal{L}^+(X)$ be the space of fully supported LPS's over X, that is, the space of finite sequences $\sigma = (\mu_0, \ldots, \mu_{n-1})$, for some integer n, where $\mu_i \in \Delta(X)$ and $\bigcup_{i=0}^{n-1} \text{supp } \mu_i = X$. In addition, the measures μ_i in σ are required to be non-overlapping, that is, mutually singular. A lexicographic type structure is a type structure where $\lambda^a : T^a \to \mathcal{L}^+(S^b \times T^b)$, and similarly for b. An event E is **assumed** by type t^a of Ann if and only if there is a level j such that $\lambda^a(t^a)$ assigns probability one to the event E for all levels $k \leq j$, and assigns probability zero to the event for all levels $k > j$. Yang (2009) uses a weaker notion that allows the levels higher than j to assign positive (and strictly smaller than 1) weights to the event. The use of lexicographic beliefs is to be contrasted with our use of standard beliefs.

RCAR in BFK is characterized by the SAS and RmAR (m levels of mutual assumption) produces the IA set in a complete structure, for big enough m. Since RmcBER generates the IA set as well, it is important to know what is the relationship between RCAR and RCvBER in terms of the solution concepts they generate. The following Proposition and examples show that RCvBER generates a strict subclass of SAS, hence it is a more restrictive notion than RCAR. However, as we show in the following section, RCvBER and RCBER are always nonempty in a complete, continuous and compact structure, unlike RCAR. Let A^a and A^b be the set of Ann's and Bob's admissible strategies, respectively.

Proposition 5.

(i) Fix a type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then $\text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$ is an $SAS_{A^a \times A^b}$.

(ii) Fix an $SAS_{Q^a \times Q^b}$ $Q^a \times Q^b$. Then there is a type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$ with $Q^a \times Q^b = \text{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}^a_m \times \text{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}^b_m$.

Proof. For part (i), if $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$ is empty, then the conditions for $\mathrm{SAS}_{A^a \times A^b}$ are satisfied, so suppose that it is nonempty. By definition of event-rationality and Lemma 2, each $s^a \in Q^a = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a$ is admissible with respect to $S^a \times S^b$ and $S^a \times Q^b$.

Suppose $s^a \in Q^a$, $r^a \in \mathrm{su}_{A^b}(s^a)$ and r^a is admissible. This implies that for any t^a, $(s^a, t^a) \in \bigcap_{m=1}^{\infty} \overline{R}_m^a$ implies that supp $\mathrm{marg}_{S^b}\mu^a(t^a) \subseteq A^b$ and r^a is optimal under $v = \mathrm{marg}_{S^b}\mu^a(t^a)$ (Lemma D.2 in BFK). Because r^a is admissible we have that $(r^a, t^a) \in \overline{R}_1^a$. For each $m \geq 2$, $(s^a, t^a) \in \overline{R}_m^a$ implies that t^a has a validated belief in R_{m-1}^b. Because $\mathrm{proj}_{S^b} R_{m-1}^b \subseteq A^b$ and $r^a \in \mathrm{su}_{A^b}(s^a)$, we have that $(r^a, t^a) \in \overline{R}_m^a$ and $r^a \in Q^a$.

For part (ii) fix an $\mathrm{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ and note that for each $s^a \in Q^a$ which is admissible with respect to Q^b, there is a v with supp $v = Q^b$ under which s^a is optimal. We can choose v such that r^a is optimal under v if and only if $r^a \in \mathrm{su}_{Q^b}(s^a)$ (Lemma D.4 in BFK). Define type spaces $T^a = Q^a$, $T^b = Q^b$, with λ^a and λ^b chosen so that supp $\mu^a(s^a) = \{(s^b, s^b) : s^b \in Q^b\}$ and supp $\mu^b(s^b) = \{(s^a, s^a) : s^a \in Q^a\}$; and $\ell^a(s^a) = \{S^b\}$ and $\ell^b(s^b) = \{S^a\}$ for all s^a and s^b.

By construction and applying similar arguments as in the proof of Proposition 1, we have that $Q^a = \mathrm{proj}_{S^a} \overline{R}_1^a$ and $Q^b = \mathrm{proj}_{S^b} \overline{R}_1^b$. Moreover, each type $t^a \in Q^a$ puts positive probability only to elements in the diagonal (s^b, s^b), which consists of event-rational strategy-type pairs, hence t^a has a validated belief in$\overline{R}_1^b$. Since all types only consider the list $\{S^b\}$ as possible, we have that $\overline{R}_m^a = \overline{R}_1^a$ and $\overline{R}_m^b = \overline{R}_1^b$ for all m, by induction. Since $\mathrm{proj}_{S^a} \overline{R}_1^a \times \mathrm{proj}_{S^b} \overline{R}_1^b = Q^a \times Q^b$ we also have $Q^a \times Q^b = \mathrm{proj}_{S^a} \bigcap_{m=1}^{\infty} \overline{R}_m^a \times \mathrm{proj}_{S^b} \bigcap_{m=1}^{\infty} \overline{R}_m^b$. $\square$

In words, for a given type structure, the strategies compatible with RCvBER form a subclass of all of the SAS, and there is a class of SAS (the $Q^a \times Q^b$ sets that are $\mathrm{SAS}_{Q^a \times Q^b}$) whose strategies are compatible with RCvBER for some type structure. Because an $\mathrm{SAS}_{Q^a \times Q^b}$ $Q^a \times Q^b$ is an $\mathrm{SAS}_{A^a \times A^b}$ but the converse is not true, Proposition 5 does not provide a characterization of RCvBER. It does show, however, that RCAR, which is characterized by SAS (BFK, Proposition 8.1), is less restrictive than RCvBER.

In fact, the following game provides an example of an SAS that is not an $\mathrm{SAS}_{A^a \times A^b}$ and cannot be generated by RCvBER for any type structure. Hence, RCvBER generates a strict subclass of SAS.

	L	C	R
U	$1,1$	$2,1$	$1,1$
M	$2,2$	$0,1$	$1,0$
D	$0,1$	$4,2$	$0,0$

Note that all strategies except for R are admissible and that $\{\mathrm{U}\} \times \{\mathrm{L},\mathrm{C}\}$ is an SAS but not an $\mathrm{SAS}_{A^a \times A^b}$. The reason is that D and M are in the support of a mixed strategy (assigning weight 1/2 to each) that is equivalent to U given that Bob plays his admissible strategies L and C, but not given the set of all strategies S^b. Since D and M are not included in $\{\mathrm{U}\} \times \{\mathrm{L},\mathrm{C}\}$, this is not an $\mathrm{SAS}_{A^a \times A^b}$.

We now argue that $\{\mathrm{U}\} \times \{\mathrm{L},\mathrm{C}\}$ cannot be the outcome of RCvBER. First, note that if this were the case, the types of Ann included in RCvBER should assign zero probability to Bob playing R. Note also that U is a best response only when $Pr(\mathrm{L}) = \frac{2}{3}$ and $Pr(\mathrm{C}) = \frac{1}{3}$ and, for these conjectures, also M and D are best responses. Is it possible that M and D are excluded because types playing these strategies are not $\{\mathrm{L},\mathrm{C}\}$-rational or S^b-rational? No, because M and D are admissible with respect to both $\{\mathrm{L},\mathrm{C}\}$ and S^b. Hence, under RCvBER, for any type structure, whenever U is included, M and D are included as well.

In the following game all strategies are admissible, hence an SAS is equivalent to an $\mathrm{SAS}_{A^a \times A^b}$.

	L	C	R
U	$1,1$	$2,1$	$1,1$
M	$2,2$	$0,1$	$1,5$
D	$0,1$	$4,2$	$0,0$

The same arguments show that RCvBER cannot produce $\{\mathrm{U}\} \times \{\mathrm{L},\mathrm{C}\}$ which is both an SAS and an $\mathrm{SAS}_{A^a \times A^b}$ but not an $\mathrm{SAS}_{Q^a \times Q^b}$. Hence, we cannot have a tighter characterization in terms of Proposition 5.

As a last comparison note that, from the proof of Proposition 4, a type of Ann that is event-rational and has $(m+1)$-th order validated belief of event-rationality in a complete type structure, necessarily has the sets S_0^b, S_1^b, ..., S_m^b in the type's tie-breaking list. This gives the intuition behind how RCvBER generates the IA set. In comparison, in BFK a type t^a of Ann that is rational and satisfies $(m+1)$-th order assumption of rationality in a complete type structure, necessarily satisfies

$$\forall k \leq m, \exists j, \bigcup_{i \leq j} \text{supp } \mu_i = S_k^b$$

where $(\mu_0, ..., \mu_{n-1})$ is the list of marginals over S^b associated with type t^a.

7 Possibility Results for RCBER and RCvBER

Since the games are assumed to be finite, Propositions 2 and 4 suggest that RmBER and RmvBER generate the $S^\infty W$ and IA sets, respectively, for m large enough. However, an epistemic criterion for $S^\infty W$ and IA has to be the same across all games and therefore independent of m. Below we show that RCBER and RCvBER are nonempty whenever the type structure is complete, continuous and compact (and recall that the universal type structure (Mertens and Zamir (1985) and Appendix B) satisfies these properties), hence providing an epistemic criterion for $S^\infty W$ and IA.

Proposition 6. *Fix a complete, continuous and compact type structure $\langle S^a, S^b, L^a, L^b, T^a, T^b, \lambda^a, \lambda^b \rangle$. Then RCBER and RCvBER are nonempty.*

Proof. First note that from Propositions 2 and 4, the sets $R_m^a \times R_m^b$ and $\overline{R}_m^a \times \overline{R}_m^b$ are non-empty for each $m \geq 1$.

We first show that R_1^a is closed. Note that T^a is compact. For any sequence (s_n^a, t_n^a) in R_1^a, we have $s_n^a \in BR(v_n^a)$, where $v_n^a = \text{marg}_{S^b} \mu^a(t_n^a)$. If $(s_n^a, t_n^a) \to (s^a, t^a)$, then $v_n^a \to v^a = \text{marg}_{S^b} \mu^a(t^a)$, implying that $s^a \in BR(v^a)$. Also, because S^a is finite, we have $s^a = s_n^a$ for large n, so $s^a \in BR^a(v_n^a)$. Further, because S^b is finite, we can choose a subsequence with supp $v_n^a =$ supp v_k^a for all indices n, k and a fortiori supp $v^a \subset$ supp v_n^a. Let σ^a satisfy $\sigma^a \sim_{\text{supp } v^a} s^a$. If supp $v^a =$ supp v_n^a we have $\sigma^a \sim_{\text{supp } v_n^a} s^a$. Hence, for each $F_i \in \ell^a(t^a)$, there exists v_i with support equal to $F_i \setminus \text{supp } v^a$, such that $\pi^a(s^a, v_i) \geq \pi^a(\sigma^a, v_i)$. If supp $v^a \neq$ supp v_n^a, then because $s^a \in BR^a(v_n^a)$ and $\sigma^a \sim_{\text{supp } v^a} s^a$, it must be that there exists $\eta \in \Delta(S^b)$ with $\pi^a(s^a, \mu) \geq \pi^a(\sigma^a, \eta)$ and supp $\eta =$ supp $v_n^a \setminus$ supp v^a (η can be taken as the conditional of v_n^a on supp $v_n^a \setminus$supp v^a). Now put $\eta' = \alpha\eta + (1-\alpha)v_i$ for some $\alpha \in (0,1)$, note that supp $\eta' = F_i \setminus$ supp v^a and that $\pi^a(s^a, \eta') \geq \pi^a(\sigma^a, \eta')$. That is, $(s^a, t^a) \in R_1^a$, so it is a closed subset of the compact space $S^a \times T^a$.

Consider $R_2^a = R_1^a \cap [S^a \times B^a(R_1^b)]$, and pick a convergent sequence (s_n^a, t_n^a) therein, with limit (s^a, t^a). Because R_1^b is closed and λ^a is continuous, we have $\limsup_{t_n^a \to t^a} \mu^a(t_n^a)(R_1^b) \leq \mu^a(t^a)(R_1^b)$. Hence $\mu^a(t^a)(R_1^b) = 1$ because $\mu^a(t_n^a)(R_1^b) = 1$ for every n. Also, event-rationality follows from an argument similar to the argument above, and we conclude that R_2^a is compact. Inductively, R_m^a is compact for all m. It follows that $\bigcap_{m\geq 1} R_m^a \neq \emptyset$ because the family $\{R_m^a\}_{m\geq 1}$ has the finite intersection property: for any finite list $\{m_1, \ldots, m_K\}$ of positive numbers, let $m_{\overline{k}}$ be the largest. Then we know that $R_{m_{\overline{k}}}^a \neq \emptyset$ and it is included in $\bigcap_{k=1}^K R_{m_k}^a$.

We also have compactness of the sets $\overline{R}_m^a$. Pick a sequence (s_n^a, t_n^a) in $\overline{R}_m^a$ converging to (s^a, t^a), and without loss of generality focus on a subsequence with $\ell^a(t_n^a) = \ell^a(t_k^a)$ for all n, k. It must then be that $\ell^a(t_n^a) = \ell^a(t^a)$. Repeat the argument in the first paragraph of the proof to conclude that (s^a, t^a) is event-rational because (s_n^a, t_n^a) is event-rational for each n, and $\mathrm{proj}_{S^b} \overline{R}_{m-1}^b \in \ell^a(t^a)$, so $(s^a, t^a) \in \overline{R}_m^a$. Hence we have a nested sequence of non-empty compact spaces, so by the finite intersection property, we have $\bigcap_{m\geq 1} R_m^a \neq \emptyset$.

The same arguments apply to b.

□

8 Conclusion

Let us summarize the contributions of the paper. (1) We define a new notion of rationality, named *event-rationality*, and provide preference basis for it. The preferences of event-rational players are represented by a pair (μ, ℓ), where μ is a probability measure and ℓ is a set of events used for breaking ties. We require that the set of all strategies of the opponent is a member of ℓ, and as a result obtain that event-rational players play admissible strategies. (2) We define and provide decision theoretic foundations for a new notion of "believing", named *validated belief*, which relates to the preference representation of event-rationality. (3) We provide epistemic conditions for two well-known solution concepts in game theory, $S^\infty W$ and IA. We do so by constructing the set of states where "rationality and common belief of rationality" obtain, using event-rationality as the notion of rationality, and (for the IA case) validated belief as the notion of belief. The epistemic characterization of IA solves a well-known and much-studied problem in a novel way. And, importantly, it does not require the use of incomplete or discontinuous type structures. (4) We develop new solution concepts, HAS and HIA, that are induced by RCBER and RCvBER, respectively, for any type structure, not necessarily complete. (5) Finally, we show that RCvBER can be used to justify a strictly smaller class of solutions than BFK's RCAR, thus showing that RCvBER and RCAR are not merely isomorphic conditions written in two different languages.

A Preference Basis

We develop below preference foundations for event rationality and validated beliefs, using the idea that a decision maker is represented by a list of preferences. Alternative foundations can be provided. Indeed, in an ealier version of this paper we provided foundations based on Manzini and Mariotti (2007).

Let Ω be a state space and $\mathcal{A}$ the set of all measurable functions from Ω to $[0,1]$. For simplicity, assume that Ω is finite (modulo technical details, the considerations below carry through in a more general state space). A decision maker has preferences over elements of $\mathcal{A}$. We assume that the outcome space $[0,1]$ is in utils. That is, all preferences considered below agree on constant acts over an outcome space, so the Bernoulli indices are uniquely defined and omitted from the analysis that follows. For $x, y \in \mathcal{A}$, $0 \leq \alpha \leq 1$, $\alpha x + (1-\alpha)y$ is the act that at ω gives payoff $\alpha x(\omega) + (1-\alpha)y(\omega)$. Unless otherwise noted, we assume that a preference relation $\succsim$ satisfies completeness, transitivity, independence and has an expected utility representation.

Definition 15. *$x \succsim_E y$ if for some $z \in \mathcal{A}$, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$.*

Note that for preferences satisfying the aforementioned axioms, $(x_E, z_{\Omega\setminus E}) \succsim (y_E, z_{\Omega\setminus E})$ holds for all z if it holds for some z. An event E is **Savage null** if $x \sim_E y$ for all $x, y \in \mathcal{A}$. For a given $\succsim$, the set $N(\succsim) \subset \Omega$ denotes the union of all non Savage null events according to $\succsim$.

Fix a game and the resulting set of available acts $\mathcal{B}$. An act $x \in \mathcal{B}$ is **event-rational** if there exist a preference $\succsim$ and a list $\ell = \{F_1, ..., F_k\}$, with $F_i \subset \Omega$ for $i = 1, ..., k$ such that

- $x \succsim y$ for every $y \in \mathcal{B}$,
- for each $F_i \in \ell$ with $F_i \setminus N(\succsim) \neq \emptyset$ and act $y \in \mathcal{B}$ with $x(\omega) = y(\omega)$ for all $\omega \in N(\succsim)$, there exists a preference $\succsim'$ with $N(\succsim') = F_i \setminus N(\succsim)$ such that $x \succsim' y$,
- $\Omega \in \ell$.

Therefore, the definition of event-rationality is identical to that of the main text.

Consider a decision maker represented by a list of preferences $\{\succsim_i\}_{i=0}^k$ with $N(\succsim_i) \cap N(\succsim_0) = \varnothing$ for $i = 1, ..., k$ and $N(\succsim_i) = \Omega \setminus N(\succsim_0)$ for some i.[12] The interpretation is that $N(\succsim_0)$ is the theory of the decision maker, and the list $\{N(\succsim_i)\}_{i=1}^k$ represent probability-zero *gedankenexperiments*, used to break ties. Formally, given a list of preferences $\{\succsim_i\}_{i=0}^k$ satisfying the aforementioned two properties we define an induced preference relation over acts, $\succsim^c$, as follows:

Definition 16. *$x \succsim^c y$ if and only if either*

- *$x \succsim_0 y$ and $x \neq y$ on $N(\succsim_0)$ or*
- *$x = y$ on $N(\succsim_0)$ and $x \succsim_i y$ for $i = 1, ..., k$.*

Note that $\succsim^c$ is incomplete but transitive. An act x is $\succsim^c$-rational if $x \succsim^c y$ for every $y \in \mathcal{B}$.

Proposition 7. *An act x is $\succsim^c$-rational if and only if it is event-rational.*

Proof. By definition, if x is $\succsim^c$-rational, then it is event-rational under $\succsim = \succsim_0$ and $\ell = \{F_1, ..., F_k\}$, with $F_i = N(\succsim_i) \cup N(\succsim_0)$ for $i = 1, ..., k$.

Conversely, let x be event-rational under $\hat{\succsim}$ and $\ell = \{F_1, ..., F_k\}$. If $x \neq y$ on $N(\hat{\succsim})$, then $x \succsim^c y$ using $\succsim_0 = \hat{\succsim}$. So let us focus on acts in $C = \{y \in \mathcal{B} : y = x \text{ on } N(\hat{\succsim})\}$. Let $m = \#\Omega \setminus N(\hat{\succsim})$, and note that the set C can be identified as a convex in $[0,1]^m$, with $x \in C$. For each $i = 1, ..., k$ where $E_i = F_i \setminus N(\hat{\succsim}) \neq \emptyset$, let $B_i = \{r \in \mathbb{R}_+^m : r|_{E_i} \gg x|_{E_i}\}$, where $x|_{E_i}$ denotes the vector x restricted to states in E_i. Note that $B_i \cap C = \emptyset$, because otherwise there would exist an act y that is outcome-equivalent to x and strictly preferred to x for any preference $\succsim'$ with $N(\succsim') = E_i$, contradicting event-rationality of x. Because B_i is also convex, by the separating hyperplane theorem there exists $\alpha_i \in \mathbb{R}^m$ with $\alpha_i \cdot r > \alpha_i \cdot y$ for all $r \in B_i$ and $y \in C$. Take $r^\varepsilon \in \mathbb{R}_+^m$ with $r^\varepsilon(\omega) = x(\omega)$ for $\omega \notin E_i$ and $r^\varepsilon(\omega) = x(\omega) + \varepsilon$ for $\omega \in E_i$ and $\varepsilon > 0$. Then $r^\varepsilon \in B_i$. Letting $\varepsilon \to 0$, we have $r^\varepsilon \to x$ and we obtain $\alpha_i \cdot x \geq \alpha_i \cdot y$ for every $y \in C$.

Also, α_i can be chosen to satisfy $\alpha_i(\omega) > 0$ only if $\omega \in E_i$. Otherwise, say that $\alpha_i(\omega') > 0$ and $\omega' \notin E_i$. If $y(\omega') = 0$ for every act in $\mathcal{B}$, then $\alpha_i(\omega')$ can be set equal to zero without

[12] One can think of conditional preferences, as in Luce and Krantz (1971), Fishburn (1973) and Ghirardato (2002).

loss. If $x(\omega')=0$ and there exists $y\in C$ with $y(\omega')>0$, then it cannot be the case that $F_i=\{\omega'\}$ for any $i=1,...,k$. So set $y(\omega)=x(\omega)$ for every $\omega\neq\omega'$ and $y(\omega')>x(\omega')$, with $y\in C$. Such a y exists because $E_i\neq\Omega\setminus N(\hat{\succsim})$ (if it was equal, then ω' would not exist) and there is no F_i equal to $\{\omega'\}$. Then $\alpha_i\cdot r^\varepsilon>\alpha_i\cdot y$, for the r^ε constructed above. But as $\varepsilon\to0$, $r^\varepsilon\to x$ and $\alpha_i\cdot x<\alpha_i\cdot y$ by construction. This contradicts $\alpha_i\cdot r^\varepsilon>\alpha_i\cdot y$ for all ε. In the case that $x(\omega')>0$, change the r^ε above by having $r^\varepsilon(\omega')=0$, while keeping the other values. Then as $\varepsilon\to0$, we must get $\alpha_i\cdot r^\varepsilon<\alpha_i\cdot x$, another contradiction. So the support of α_i is contained in E_i.

Moreover, because for each $y\in C$ there exists $\succsim'$ with $N(\succsim')=E_i$ and $x\succsim' y$, it must be that $\alpha(\omega)>0$ if $\omega\in E_i$. If not, then there is $\omega'\in E_i$ with $\alpha_i(\omega')=0$, and there is no other α_i' with $\alpha_i'(\omega')>0$ that would separate B_i and C. Now take the original r^ε and $y\in C$ with $y(\omega')>x(\omega')$. Such a y must exist, for otherwise there would exist the required α_i'. But there is no $\succsim'$ with $N(\succsim')=E_i$ and $x\succsim' y$, a contradiction. So it must be that $\alpha_i(\omega)>0$ if and only if $\omega\in E_i$.

Normalizing α_i yields a probability distribution ν_i with $\operatorname{supp}\nu_i=E_i$ for which x is a better response than any $y\in C$. Let $\succsim_i$ be the preference relation represented by the underlying Bernoulli index and ν_i. The construction above is true for every $i=1,...,k$. Setting $\succsim_0=\hat{\succsim}$ and collecting the list $\{\succsim_0,\succsim_1,...,\succsim_k\}$ it follows that x is $\succsim^c$-rational. $\square$

In what follows, for ease of notation, we use $N_i=N(\succsim_i)$ for $i=0,...,k$, $x\succ_{iE}y$ to denote that x is preferred to y according to $\succsim_i$ conditional on E (according to Definition 15), and $x=_{0E}y$ to denote that $x(\omega)=y(\omega)$ for all $\omega\in N_0\cap E\neq\emptyset$. The notions of beliefs we use in the main text are as follows.

Definition 17. *Event E is **believed under** $\succsim^c$ if $N_0\subset E$.*

Definition 18. *Event E has a **validated belief under** $\succsim^c$ and i if $E=N_0\cup N_i$.*

In words, the decision maker believes an event E if she believes it according to her theory; and she has a validated belief in it if it is equal to the union of N_0 and some N_i. Note that it may well be that $i=0$, so the decision maker may have a validated belief in the event $E=N_0$. Note that in the text we "validated" a belief with events that describe strategies only. Here we do not make this distinction for ease of exposition. It is straightforward to consider a product state space $\Omega=\Omega_1\times\Omega_2$ and define belief for events on Ω and validated beliefs as those that are validated by the projection of an N_i to Ω_1.

We now define a notion of conditional $\succsim^c$-preference that is consistent with tie-breaking ideas.

Definition 19. *Say that $x \succ_E^c y$ under i if*

- $x \succ_{0E} y$ *or*
- $x =_{0E} y$, $x \succ_{iE} y$ *and* $x \succsim_j y$ *for every* $j \neq i$.

Say that $x \succ_E^c y$ if $x \succ_E^c y$ for some i. Note that $x \succ_E^c y$ under i and $x =_{0E} y$ necessarily mean that $i > 0$.

Definition 20. *An event E is **nontrivial under** $\succsim^c$ and i if*

- *there is a pair x, y with $x \succ_E^c y$ under i, and*
- *if $\omega \in E$ is such that there is no pair x, y with $x \succ_\omega^c y$, then there is a pair x, y with $x = y$ on N_0 such that $x \succ_{E(\omega)}^c y$ under i, where $E(\omega) = E \cap (N_0 \cup \{\omega\})$.*

Definition 21. *An event E satisfies **strict determination under** $\succsim^c$ and i if for all x, y, $x \succ_E^c y$ under i implies $x \succ^c y$.*

The following Lemma characterizes validated belief with respect to nontriviality and strict determination.

Lemma 3. *There exists i such that E has a validated belief under $\succsim^c$ and i if and only if it is nontrivial and satisfies strict determination under $\succsim^c$ and i.*

Proof. By nontriviality, $E \cap N_0 \neq \varnothing$, for otherwise there would exist no pair x, y with $x \succ_E^c y$. Assume by way of contradiction that there exists $\hat{\omega} \in N_0 \setminus E$. Also, let $\omega' \in E \cap N_0$. Set $x(\omega') = 1$ and zero otherwise, and set

$$y(\omega) = \begin{cases} a & \text{if } \omega = \hat{\omega} \\ b & \text{if } \omega = \omega' \\ 0 & \text{otherwise} \end{cases}$$

where $a > \frac{v_0(\omega')(1-b)}{v_0(\hat{\omega})}$, $0 < b < 1$, and v_0 is the conjecture associated with $\succsim_0$. Then, conditional on E, the payoff of x is equal to 1 whereas the payoff of y is $b < 1$, so $x \succ_E^c y$; But the unconditional payoff of x is equal to $v_0(\omega')$ whereas the payoff of y is $av_0(\hat{\omega}) + bv_0(\omega')$, so $y \succ^c x$, contradicting strict determination. Hence $N_0 \subset E$. Therefore, if for all $\omega \in E$ there exists a pair x, y with $x \succ^c_\omega y$, then $E \subset N_0$, and we conclude that $E = N_0 \cup N_i$, with $i = 0$.

If there is $\omega \in E$ for which there is no pair x, y with $x \succ^c_\omega y$, then $\omega \notin N_0$. By nontriviality, there is a pair x, y with $x = y$ on N_0 with $x \succ^c_{E(\omega)} y$ under i, meaning that $x \succ_{iE(\omega)} y$, which in turn means that $\omega \in N_i$ and $i \neq 0$. Hence we must have $E \subset N_0 \cup N_i$. Similarly to above, assume by way of contradiction that there exists $\hat{\omega} \in N_i \setminus E$. Also, let $\omega' \in E \cap N_i$. Construct x and y as follows: $x = y$ on N_0, and on $\Omega \setminus N_0$ x and y are as above, with $a > \frac{v_i(\omega')(1-b)}{v_i(\hat{\omega})}$. Strict determination is again violated, so we must have $N_0 \cup N_i \subset E$, and we conclude that $E = N_0 \cup N_i$ with $i > 0$.

Conversely, assume that $E = N_0 \cup N_i$ for some i. Let $x = 1$ on N_0, 0 otherwise and $y(\omega) = 0$ for every ω. Then $x \succ^c_0 y$ and $x \succ^c_E y$ under i. For the second condition, if $i = 0$, then $E = N_0$ and there does not exist $\omega \in E$ such that there is no pair x, y with $x \succ^c_\omega y$. If $i \neq 0$, pick $\omega \in N_i$ (so $\omega \notin N_0$). Set $x = y$ on N_0, $x(\omega) = 1$, $y(\omega) = 0$ and $x = y = 0$ elsewhere. Then $x \succ^c_{E(\omega)} y$, so nontriviality is satisfied.

Finally, let $x \succ^c_E y$ under i. If $x \succ_{0E} y$ then $x \succ_0 y$, implying that $x \succ^c y$. If $x =_{0E} y$, $x \succ_{iE} y$ and $x \succsim_j y$ for every $j \neq i$, then $x = y$ on N_0, $x \succ_i y$ and $x \succsim_j y$ for every $j \neq i$, which again means that $x \succ^c y$. So strict determination is satisfied.

□

Corollary 1. *An event E is believed under $\succsim^c$ if and only if it satisfies strict determination under $\succsim^c$ and $i = 0$ and there exists a pair x, y with $x \succ^c_E y$ under $i = 0$.*

B Type Spaces

We now show that the by now standard construction of all hierarchies of beliefs about beliefs generates a complete and continuous type structure. Because the types consistent with event-rationality are mapped to both probability measures and lists, we need to adapt the standard construction. One route is to follow Epstein and Wang (1995) and work with more general beliefs about beliefs. Another route, followed bellow, is to construct an auxiliar complete, continuous and compact type structure, using the standard construction, and then use it to construct the desired type structure.

Let $\Delta^*(X \times L^i)$ be the space of all probability measures over $X \times L^i$ (endowed with the weak* topology) for which the marginal on L^i is a mass point, for $i = a, b$.

Let $\Omega_1^a = S^b \times L^b$ and $T_1^a = \Delta^*(S^b \times L^b)$. Inductively set $\Omega_{k+1}^a = S^b \times L^b \times T_k^b$ where

$$T_{k+1}^a = \{(\mu_1^a, ..., \mu_k^a, \mu_{k+1}^a) \in T_k^a \times \Delta^*(\Omega_{k+1}^a) : \mathrm{marg}_{\Omega_k^a} \mu_{k+1}^a = \mu_k^a\}.$$

Likewise for b. Then the standard arguments in the literature show the existence of compact spaces T_*^a and T_*^b, with T_*^a homeomorphic to $\Delta^*(S^b \times T_*^b \times L^b)$ and T_*^b homeomorphic to $\Delta^*(S^a \times T_*^a \times L^a)$.[13] In fact, let T_*^a be the projective limit of the spaces $(T_k^a)_{k=1}^\infty$. T_*^a is compact as it is a product of compact spaces. Construct T_*^b similarly. Then Theorem 8 in Heifetz (1993) shows that for each tower $(\mu_k^a)_{k=1}^\infty$ there exists $\mu^a \in \Delta(S^b \times L^b \times T_*^b)$ with $\mathrm{marg}_{\Omega_k^a} \mu^a = \mu_k^a$ for all $k \geq 1$. In particular, the marginal of μ^a on L^b is a mass point, so $\mu^a \in \Delta^*(S^b \times L^b \times T_*^b)$. Conversely each $\mu^a \in \Delta^*(S^b \times L^b \times T_*^b)$ gives rise to a tower $(\mu_k^a)_{k=1}^\infty$ given by the list of marginals. So there is a bijection $\lambda_*^a : T_*^a \to \Delta^*(S^b \times L^b \times T_*^b)$. Theorem 9 in Heifetz (1993) ensures that λ_*^a is a homeomorphism, likewise for b. So we have constructed a complete, continuous and compact auxiliar type structure

$$\langle S^i, L^i, T_*^i, \lambda_*^i \rangle_{i \in \{a,b\}}$$

with $\lambda_*^i : T_*^i \to \Delta^*(S^j \times T_*^j \times L^j)$ for $j \neq i = a, b$. Note that $\lambda_*^i(t_*^i) = \mu(t_*^i) \otimes \delta_{\ell(t_*^i)}$ where δ_x is the point mass at x.

Now set $T^i = T_*^i$ (carrying the same topology, so T^i is compact Hausdorff) and $\lambda^i(t_*^i) = (\mu(t_*^i), \ell(t_*^i))$, for $i = a, b$. The assignment $\lambda_*^i \mapsto \lambda^i$ is a bijection and preserves continuity: λ^i is continuous if and only if λ_*^i is continuous. Indeed, let $t_\alpha^i \to t^i$ in T^i. This is a converging net in T_*^i, so $\lambda_*^i(t_\alpha^i) \to \lambda_*^i(t^i)$, or $\mu(t_\alpha^i) \otimes \delta_{\ell(t_\alpha^i)} \to \mu(t^i) \otimes \delta_{\ell(t^i)}$. But $\delta_{\ell(t_\alpha^i)} \to \delta_{\ell(t^i)}$ in the weak* topology if and only if $\ell(t_\alpha^i) \to \ell(t^i)$. So $(\mu(t_\alpha^i), \ell(t_\alpha^i)) \to (\mu(t^i), \ell(t^i))$, or $\lambda^i(t_\alpha^i) \to \lambda^i(t^i)$, for $i = a, b$. A similar argument establishes that λ_*^i is continuous if λ^i is continuous. Moreover, λ^i is injective and surjective. Hence it is a homeomorphism, as a continuous bijection between compact Hausdorff spaces. So the type structure

$$\langle S^i, L^i, T^i, \lambda^i \rangle_{i \in \{a,b\}}$$

with $\lambda^i : T^i \to \Delta(S^j \times T^j) \times L^j$ for $j \neq i = a, b$ just constructed is complete, continuous and compact.

It is important to emphasize a conceptual point here. The two players form beliefs about beliefs about what is relevant for rational choices. That is, Ann has beliefs over $S^b \times L^b$, and

[13] See for instance Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Heifetz (1993).

these beliefs are given by a conjecture over S^b and a list $\ell \in L^b$ (or, equivalently, a point mass over L^b.) What is relevant for event-rational choices is precisely the conjecture and the list. But Ann does not know what Bob's beliefs are, and the hierarchies of beliefs about beliefs constructed above yield a type structure as the one we use in the paper.

B.1 Lists over Types

We argued in the text that lists over strategies suffice for the analysis. Indeed, it is redundant to include subsets of types in the tie-breaking lists, as types do not play any role in breaking ties. Also, provided that we consider a rich list of subsets of types, such lists would not interfere in the constructions in the text that used validated beliefs. Let us now show how to obtain a type structure with rich lists over strategies and types from a given type structure.

Let the type structure $\langle S^i, L^i, T^i, \lambda^i \rangle_{i\in\{a,b\}}$ be given. For $i \neq j = a, b$, let $\mathcal{F}(T^i)$ denote the space of all closed subsets of T^i, endowed with the Fell topology.[14] Say $\ell^i(t^i) = \{E_1, ..., E_k\}$, with $E_r \subset S^j$ for $r = 1, ..., k$. Let $E_r = \{s^j_1, ..., s^j_m\}$ and construct $\mathcal{E}_r = \{(\{s^j_1\} \times K, ..., \{s^j_m\} \times K') : (K, ..., K') \in (\mathcal{F}(T^j))^m\}$, for $r = 1, ..., k$, where $(\mathcal{F}(T^j))^m$ denotes the product of m copies of $\mathcal{F}(T^j)$. Note that $\mathcal{E}_r$ is compact whenever T^j is Hausdorff. Finally, put $\hat{\ell}^i(t^i) = \{\mathcal{E}_1, ..., \mathcal{E}_k\}$ as the extended list. Repeat the procedure for all t^i and $i = a, b$, to construct the type structure

$$\langle S^i, \hat{L}^i, T^i, \hat{\lambda}^i \rangle_{i\in\{a,b\}}$$

where $\hat{\lambda}^i = (\mu^i, \hat{\ell}^i)$ and $\hat{L}^i$ is the space of extended lists (as the one constructed above) of subsets of $S^i \times T^i$.

Now, for any closed subset $F \subset S^j \times T^j$, we have

$$F \in \hat{\ell}^i(t^i) \Leftrightarrow \mathrm{proj}_{S^j} F \in \ell^i(t^i).$$

That is, extended lists do not interfere with statements about validated beliefs. Extended lists do not interfere with breaking ties either. So the arguments in the text apply to the corresponding type structure with extended lists with no change (other than notation).

[14] See for instance Molchanov (2005) for definitions of topologies on spaces of subsets. The nice feature of the Fell topology is that $\mathcal{F}(T^i)$ is compact whenever T^i is Hausdorff. When T^i is compact metric, the Fell topology coincides with the standard Hausdorff metric topology.

References

Elchanan Ben-Porath. Rationality, nash equilibrium, and backward induction in perfect information games. *Review of Economic Studies*, 64:23–46, 1997.

Douglas B. Bernheim. Rationalizable strategic behavior. *Econometrica*, 52:1007–1028, 1984.

Larry Blume, Adam Brandenburger, and Eddie Dekel. Lexicographic probabilities and choice under uncertainty. *Econometrica*, 59:61–79, 1991.

Tilman Börgers. Weak dominance and approximate common knowledge. *Journal of Economic Theory*, 64:265–276, 1994.

Adam Brandenburger. Lexicographic probabilities and iterated admissibility. In Partha Dasgupta, Douglas Gale, Oliver Hart, and Eric Maskin, editors, *Economic Analysis of Markets and Games*, pages 282–290. MIT Press, 1992.

Adam Brandenburger. On the existence of a "complete" possibility structure. In Nicola Dimitri, Marcello Basili, and Itzhak Gilboa, editors, *Cognitive Processes and Economic Behavior*, pages 30–34. Routledge (London), 2003.

Adam Brandenburger and Eddie Dekel. Hierarchies of beliefs and common knowledge. *Journal of Economic Theory*, 59:189–198, 1993.

Adam Brandenburger, Amanda Friedenberg, and H. Jerome Keisler. Admissibility in games. *Econometrica*, 76(2):307–352, 2008.

Eddie Dekel and Drew Fudenberg. Rational behavior with payoff uncertainty. *Journal of Economic Theory*, 52:243–267, 1990.

Larry Epstein and Tan Wang. Beliefs about beliefs without probabilities. *Econometrica*, 64 (6):1343–1373, 1995.

Peter C. Fishburn. A mixture-set axiomatization of conditional subjective expected utility. *Econometrica*, 41(1):1–25, 1973.

Paolo Ghirardato. Revisiting savage in a conditional world. *Economic Theory*, 20:83–92, 2002.

Aviad Heifetz. The bayesian formulation of incomplete information - the non-compact case. *International Journal of Game Theory*, 21:329–338, 1993.

H. Jerome Keisler and Byung Soo Lee. Common assumption of rationality. Mimeo, 2011.

Elon Kohlberg and Jean-Francois Mertens. On the strategic stability of equilibria. *Econometrica*, 54:1003–1037, 1986.

Duncan Luce and Howard Raiffa. *Games and Decisions*. Wiley, New York, N, 1957.

R. D. Luce and D. Krantz. Conditional expected utility. *Econometrica*, 39(2):253–271, 1971.

Paola Manzini and Marco Mariotti. Sequentially rationalizable choice. *American Economic Review*, 97(5):1824–1839, 2007.

Jean-Francois Mertens and Shmuel Zamir. Formulation of bayesian analysis for games with incomplete information. *International Journal of Game Theory*, 14:1–29, 1985.

Ilya Molchanov. *Theory of Random Sets - (Probability and its Applications*. Springer, London, 2005.

David G. Pearce. Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52:1029–1050, 1984.

Larry Samuelson. Dominated strategies and common knowledge. *Games and Economic Behavior*, 4(2):284–313, April 1992.

Dale Stahl. Lexicographic rationalizability and iterated admissibility. *Economics Letters*, 47 (2):155–159, February 1995.

T. Tan and S. Werlang. The bayesian foundation of solution concepts of games. *Journal of Economic Theory*, 45:370–391, 1988.

Abraham Wald. Contributions to the theory of statistical estimation and testing hypotheses. *The Annals of Mathematical Statistics*, 10(4):239326, 1939.

Chih-Chun Yang. Weak assumption and admissibility in games. Mimeo, University of Rochester, 2009.