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Abstract

This paper revisits the question whether capital in a competitive equilibrium is overaccumulated from the perspective of the social planner in an incomplete market economy with risky human capital accumulation. As in Dávila, Hong, Krusell and Ríos-Rull (2012), we consider a constrained social planner who cannot complete markets, but can improve welfare by only internalizing how individual allocations affect prices. In a standard incomplete market economy with exogenous labor income shocks, Dávila et al. (2012) show that the pecuniary externalities tend to imply that capital in a competitive equilibrium is underaccumulated in economies with realistically high wealth inequality. We analytically show that by introducing risky *human capital accumulation* to this standard incomplete market model, the implication of pecuniary externalities can be overturned — the capital-labor ratio in a competitive equilibrium is likely to be too high from the perspective of the constrained planner. However, a quantitative investigation shows that an economy *with human capital* calibrated to match the inequalities of both wealth *and* earnings of the US still shows a capital-labor ratio lower than the optimal ratio. Introducing human capital does not overturn the quantitative result of Dávila et al. (2012) — underaccumulation of capital, but qualitatively, the result can be reversed.

Keywords: pecuniary externalities, constrained efficiency, human capital

JEL Classification Codes: D52, D60, H20, J24

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1 Introduction

In a standard incomplete market model with uninsurable idiosyncratic shocks, it is well known that uninsured risk leads to precautionary saving. Thus, a competitive equilibrium exhibits overaccumulation of capital in comparison with the first best planner's allocation — the solution to the social planner's problem who maximizes welfare subject to resource constraint only. From the perspective of the constrained planner who cannot use transfers to complete the market, however, precautionary saving in a competitive equilibrium does not directly imply an overaccumulation of capital because the constrained planner also has an incentive to command precautionary saving in the incomplete market. The only difference between the household and the constrained planner is that the planner considers how the allocation changes prices, and in turn, how the change in prices can improve the allocation of risk and the redistribution of income (so called pecuniary externalities), while a household cannot. Thus, it is the implication of the pecuniary externalities that determines the over or underaccumulation of capital compared to the constrained efficient allocation. This paper analyzes and quantifies these pecuniary externalities in an incomplete market model with endogenous human capital accumulation.

Recently, Dávila et al. (2012) (hereafter DHKR) analyzed the pecuniary externalities in a standard incomplete market model with exogenous uninsured idiosyncratic labor income shocks and its implication on whether capital in a competitive equilibrium is higher than in the constrained efficient allocation. DHKR show that in a standard incomplete market model with exogenous labor shock, the answer depends on the magnitude of two channels of pecuniary externalities, which have opposite effects on the level of capital. The first channel is the insurance effect. Since the only risk in the economy is labor income shock, a lower wage and a higher interest rate scale down the stochastic part of the income, implying that a decrease in capital can improve the welfare of households by changing prices in these directions. The second channel is the redistribution effect which arises from the cross-sectional capital dispersion in the long run. A lower interest rate and a higher wage rate can improve welfare of capital-poor households, and because consumption-poor households that have relatively higher marginal utility tend to be capital poor, an increase in capital can improve social welfare. Thus, over or underaccumulation of capital may arise depending on which channel of pecuniary externality dominates. DHKR consider three different calibrations and show that with a calibration that generates realistic wealth dispersion, the redistribution channel dominates, and thus there is underaccumulation of capital in a competitive equilibrium.

In DHKR, the key mechanism that leads to a higher capital in the constrained efficient allocation relative to the competitive equilibrium — the redistribution channel — arises from wealth dispersion, which is endogenously generated in an economy. On the other hand, labor dispersion is taken as exogenous in DHKR's set up, because labor income is determined by exogenous labor shocks. In reality, however, there is also endogenous dispersion of labor productivity through accumulation of human capital. For example, Huggett et al. (2011) show that determinants of endogenous human capital dispersion — initial human capital and learning ability — are more important than idiosyncratic shocks as a source of lifetime earnings dispersion (accounting for 61.5 percent of the variation). This endogenous labor dispersion can create additional redistribution channel, which has an implication opposite to that of the redistribution channel due to wealth dispersion. That is, the welfare of the human capital-poor households can be improved with a lower wage and a higher interest rate. Since the consumption poor also tend to be human capital poor, price changes through decreasing capital might improve welfare, which is exactly the opposite of the redistribution channel arising from wealth dispersion. This redistribution channel arising from labor dispersion is not present in a DHKR economy with exogenous labor shock processes. Thus, in this paper, we endogenize human capital so that labor income is also endogenously dispersed in the long run and reinvestigate the under/overaccumulation of capital.

There is another implication of introducing human capital on pecuniary externalities — the effect of risky returns to human capital on the constrained efficient allocation. If the return to human capital is risky due to multiplicative shocks to human capital, as is often assumed in the literature on human capital, an increase in human capital investment will increase both the mean and variance of the return to human capital. This then leads to a stronger insurance channel of pecuniary externalities, suggesting that a decrease in capital may improve social welfare (by decreasing wages and increasing interest rates).

Motivated by these implications, in this paper, we first examine analytically how the introduction of risky human capital accumulation changes the implication of the pecuniary externalities — under or overaccumulation of capital, using a two-period model with a CARA-normal specification. Starting from the baseline economy with exogenous labor shocks, we proceed with three extensions. First, we show that endogenizing human capital itself does not change the conclusions of DHKR. The only difference from the exogenous labor case is that as human capital is also a choice variable, the variable of interest is not the level of capital, but the capital-labor ratio. Second, we introduce heterogeneity in initial human capital so that we can generate endogenous human capital dispersion even in a two-period model and show that the introduction of human capital can overturn the conclusion of DHKR — from too low to too high capital-labor ratio, if the additional redistribution channel through human capital inequality is large enough. Finally, we extend the model by introducing a multiplicative shock to human capital and show that a stronger insurance effect might also decrease the optimal capital-labor ratio.

We then investigate the effects of the cross-correlation between wealth and human capital and human capital inequality on the optimal capital-labor ratio since they are important determinants of the redistribution channel. Even though the correlation and inequalities are endogenous objects which depend on many primitives, this analysis can be helpful to conjecture how the choice of model and primitives affects the constrained efficient allocation. We analytically show that lower correlation between wealth and human capital and higher human capital inequality tend to imply that the capital-labor ratio in a competitive equilibrium is too high from the perspective of the constrained planner, and a simple numerical example shows that the effect of the correlation can be quantitatively significant. Then, we discuss how this result can be used to conjecture the effect of more fundamental factors (such as primitive parameters and modeling specification) on the constrained efficient allocation.

In the final step, using an infinite horizon model, we quantitatively examine whether DHKR's conclusion in an economy with exogenous labor process — the underaccumulation of capital in an economy that generates high wealth inequality — is overturned by introducing human capital. We find that even though we introduce risky human capital accumulation, an economy calibrated to generate realistic inequalities of wealth and human capital maintains the conclusion of DHKR. That is, the steady state capital-labor ratio is still too low in a competitive equilibrium, even with the introduction of human capital. The degree of underaccumulation is smaller than that in the DHKR economy. The capital-labor ratio of the constrained optimum is 3.60 times as large as that of the first-best economy and 2.63 times larger than that of the competitive equilibrium, while in DHKR, the capital-labor ratio of the constrained efficient allocation is 8.5 times larger than that in the deterministic economy and 3.65 times larger than that in a competitive equilibrium.

This paper is related to the literature on the constrained efficiency/ineffciency of allocation in a competitive equilibrium with uninsured shocks. After Diamond (1967) first noted the possibility of constrained inefficiency in a competitive equilibrium based on pecuniary externalities, examples of constrained inefficiency were provided in Hart (1975), Diamond (1980), Stiglitz (1982), Loong and Zeckhauser (1982). Recently, Dávila et al. (2012) address this constrained inefficiency issue in a standard infinite-horizon neoclassical growth model with uninsured idiosyncratic shocks, which is the workhorse of the macroeconomic model. This paper is also related to Gottardi et al. (2012) who investigate whether the introduction of linear distortionary taxes or subsidies on labor income and return from saving are welfare improving in a two-period model. While Dávila et al. (2012) focus on the pure effect of pecuniary externalities, they extend the analysis to the optimal tax or subsidy in the presence of pecuniary externalities, including the case in which there are direct insurance and redistribution provided by the given tax scheme itself as well as indirect welfare improvement through price changes.

Literature on the endogenous human capital model is also closely related to this paper. Huggett et al. (2006, 2011) develop a model with endogenous accumulation of human capital to match many features of earnings distribution, and the model of Huggett et al. (2011) is the most similar to the model specification used for this paper's quantitative investigation. The additive risk and multiplicative risk to human capital return in the theoretical analysis of our paper is exploited by Singh (2010), even though the purpose of her paper is to analyze the aggregate effect of market incompleteness in the presence of both risk.

Even though we focus on the pecuniary externalities, since these pecuniary externalities can be one of the consideration of the government with other frictions and policy tools, this paper is naturally connected to the optimal taxation analysis with endogenous human capital formation. Bohacek and Kapička (2008), da Costa and Maestri (2007), Grochulski and Piskorski (2010) Kapička (2006, ming), and Kapička and Neira (2014) study the effect of endogenous human capitla on the optimal Mirrleesian taxation, and Krueger and Ludwig (2013) and Peterman (2012) study optimal Ramsey taxation in the presence of endogenous human capital.

The rest of the paper is outlined as follows. In section 2, we analytically investigate the introduction of risky human capital with two-period CARA-normal specification. In section 3, we analyze the implication of the correlation and relative size of wealth inequality and human capital inequality on the optimal capital-labor ratio. In section 4, we quantitatively examine infinite-horizon model calibrated to the data to check the optimal level of the capital-labor ratio. We conclude in section 5.

2 Two-period model with CARA-normal specification

In this section, we theoretically analyze the effects of the introduction of human capital on the constrained efficient allocations. Specifically, we are interested in how the introduction of human capital affects the evaluation on the capital-labor ratio in a competitive equilibrium

— whether the capital-labor ratio is too high or too low relative to that of the constrained efficient allocation. We first analyze this using a two-period model with a CARA-normal specification. In this simple set up, we can clearly show through which mechanism the introduction of human capital might lead to a different conclusion from that of the standard incomplete market economy with exogenous labor shock (considered by DHKR).

2.1 Baseline: DHKR economy with exogenous labor

We use the economy of DHKR as a baseline for our analysis. In DHKR economy, there is a continuum of households with measure one. Households live only for two periods. At period 0, a household is born with initial wealth k. Let Γ denote the cross-sectional distribution of capital k across households. The ex-ante heterogeneity of k in this two-period model is assumed to approximate the endogenous dispersion of wealth in an infinite horizon model. In period 0, the household with initial wealth k will decide how much to consume c_0 and to invest in capital k'. At period 1, the household receives an idiosyncratic labor endowment,

$$L + e, \quad e \sim N(0, \sigma_e^2),$$

and consumes both the labor income and return to capital investment. The labor income is stochastic due to shock e, while the return to capital is deterministic. In sum, the budget constraints of the household with initial wealth k are:

$$\begin{aligned} c_0 + k' &= k \\ c_1(e) &= w[L + e] + rk', \end{aligned}$$

where w is the wage rate and r is the interest rate.

The household maximizes expected lifetime utility $u(c_0) + \beta E_e[u(c_1)]$ subject to the budget constraint, where the period utility takes the form of CARA:

$$u(c) = -\frac{1}{\psi}\exp(-\psi c),$$

where ψ is the coefficient of risk aversion.

Competitive firms have the same production function $F(K, L)$, which is strictly increasing and strictly concave in both arguments. Additionally, we assume that the production function exhibits a constant return to scale.

We first characterize a competitive equilibrium, and then, we will characterize the constrained efficient allocation. By comparing the capital-labor ratio, we can evaluate whether the capital in a competitive equilibrium is overaccumulated relative to the constrained efficient allocation.

2.1.1 Competitive equilibrium

We start by defining a competitive equilibrium.

Definition 1. *A competitive equilibrium consists of an allocation $(k'(k))$ and prices (r, w) such that*

(i) household chooses $k'(k)$ to maximize expected lifetime utility subject to budget constraint

(ii) $K = \int k'(k)\Gamma(dk)$

(iii) $r = F_K(K, L)$, $w = F_L(K, L)$.

Notice that under the CARA-normal specification, a household's expected utility is given by

$$\begin{aligned} E_e[u(c_1)] &= E_e[-\frac{1}{\psi}\exp(-\psi c_1)] = -\frac{1}{\psi}\exp\left(-\psi\left\{E_e[c_1] - \frac{\psi}{2}Var_e(c_1)\right\}\right), \\ \textit{where} \quad & E_e[c_1] = wL + rk', \quad Var_e(c_1) = w^2\sigma_e^2, \end{aligned}$$

and the household's expected marginal utility is given by $E_e[u_c(c_1)] = -\psi E_e[u(c_1)]$. These properties of CARA-normal specification will be useful when we characterize the constrained efficient allocation — for the additive decomposition of the pecuniary externalities into two sources. For notational simplicity, we drop the subscript e of the expectation from now on.

The first order condition of the household problem with respect to k' is

$$\begin{aligned} & u'(c_0(k)) = \beta r E[u(c_1(k))|k] \\ \Leftrightarrow \quad & \exp\left(-\psi \cdot c_0(k)\right) = \beta r \exp\left(-\psi\left\{E[c_1(k)|k] - \frac{\psi}{2}Var[c_1(k)|k]\right\}\right), \end{aligned} \tag{1}$$

which equates the marginal utility of period 0 and the expected marginal utility of period 1.

We first show that there exists a precautionary saving in a competitive equilibrium. By taking log on both sides of the equation (1), we obtain the following Euler equation,

$$E[c_1(k)|k] - c_0(k) = \frac{1}{\psi}log(\beta r) + \frac{\psi}{2}w^2\sigma_e^2,$$

where $\frac{\psi}{2}w^2\sigma_e^2 > 0$ captures precautionary saving due to idiosyncratic shock.[1] Thus, the level of capital in the competitive equilibrium is higher than that of the complete market economy. Using the Euler equation and the budget constraints of households, we also obtain the following closed form solution for saving.

$$k'(k) = \frac{1}{1+r}\left[\frac{1}{\psi}log(\beta r) + \frac{\psi}{2}w^2\sigma_e^2 - wL + k\right]$$

We notice that because $\frac{\partial k'(k)}{\partial k} = \frac{1}{1+r} > 0$, the initial heterogeneity in k generates inequality in capital holdings in period 1.

The optimality condition of the firm's problem implies that the rental prices of capital and human capital are equal to the marginal productivity of capital and human capital, respectively:

$$r = F_K(K, L), \quad w = F_L(K, L),$$

where $K = \int k'(k)\Gamma(dk)$.[2]

2.1.2 Constrained efficient allocation

We first specify the set of allocations from which the constrained planner can choose. The constrained planner can choose different level of saving $k'(k)$ for each household with different initial wealth k, while respecting all budget constraints of households and market clearing conditions and letting firms behave competitively. Because the price in the households' budget constraint is determined by the firm's optimality conditions, the period-one consumption of the household with initial wealth k and labor shock e will be determined by $c_1(k, e) = F_L(K, L)(L + e) + F_K(K, L)k'(k)$. By then assuming a utilitarian social welfare function, the constrained planner's problem is defined as follows.

$$\begin{aligned} \max_{c_0(k),c_1(k),k'(k)} \quad & \int_k [u(c_0(k)) + \beta E[u(c_1(k, e))|k]]\,\Gamma(dk) \\ s.t. \quad & c_0(k) = k - k'(k) \\ & c_1(k, e) = F_L(K, L)(L + e) + F_K(K, L)k'(k) \\ where \quad & K = \int k'(k)\Gamma(dk) \end{aligned}$$

[1] If the asset market is complete, the Euler equation will be $E[c_1(k)|k] - c_0(k) = \frac{1}{\psi}log(\beta r)$. Thus, we can see that the additional positive term $\frac{\psi}{2}w^2\sigma_e^2$ in an Euler equation of incomplete market implies higher saving $k'(k)$ due to precautionary motives.

[2] In this two-period model, we assume that depreciation rate of capital is 100% for simplicity. Later, in an infinite horizon model, we use more realistic depreciation rate of capital.

Notice that the constrained planner cannot complete the market and he has to respect the budget constraints of all households. The only difference between the constrained social planner and competitive equilibrium is that the constrained social planner also considers effects of prices on welfare, which are pecuniary externalities.

We now characterize the constrained efficient allocation focusing on its Euler equation, as in DHKR. Again, due to the CARA-normal assumption, we can rewrite the social planner's problem as follows:

$$\begin{aligned} & \max_{k'(k)} \quad \int u(k-k'(k)) + \beta E\left[u(F_L(L+e)+F_K k'(k))|k\right]\Gamma(dk) \\ = & \max_{k'(k)} \quad \int \left[-\frac{1}{\psi}\exp\left(-\psi(k-k'(k))\right) - \beta\frac{1}{\psi}\exp\left(-\psi\left\{F_L L + F_K k'(k) - \frac{\psi}{2}F_L^2\sigma_e^2\right\}\right)\right]\Gamma(dk), \end{aligned}$$

which chooses the saving of each household with initial wealth k, considering its effect on the marginal productivity of capital and labor in period 1.

Then, the first order condition with respect to $k'(k)$ is the following Euler equation with an additional term:

$$\begin{aligned} u'(c_0) &= \beta F_K E[u'(c_1)|k] + \Delta_k, \qquad (2) \\ \textit{where} \quad \Delta_k &= \beta \int E[u'(c_1)|k]\left[F_{LK}L + F_{KK}k'(k) - \psi F_L F_{LK}\sigma_e^2\right]\Gamma(dk). \end{aligned}$$

By comparing this Euler equation with (1), DHKR show that Δ_k is an additional marginal benefit of saving through the change of prices[3], which is not internalized in the household problem. Thus, Δ_k captures so-called pecuniary externalities.

As in DHKR, the sign of Δ_k is the key object of the ensuing analysis. If $\Delta_k > 0$, the constrained planner makes everyone save more than he would do in a competitive equilibrium. Thus, positive Δ_k implies an underaccumulation of capital in a competitive equilibrium relative to the constrained efficient allocation. On the other hand, negative Δ_k implies an overaccumulation of capital in a competitive equilibrium.

In order to sign Δ_k, we now decompose Δ_k.

$$\begin{aligned} \Delta_k &= \Delta_{k1} + \Delta_{k2} \\ \Delta_{k1} &= -\psi F_L F_{LK}\sigma_e^2 \beta \int E[u'(c_1)|k]\Gamma(dk) \\ \Delta_{k2} &= \beta \int E[u'(c_1)|k]\left[F_{LK}L + F_{KK}k'(k)\right]\Gamma(dk) \\ &= F_{KK}K\beta \int E[u'(c_1)|k]\left[\frac{k'(k)}{K} - 1\right]\Gamma(dk) \quad (\text{using } F_{LK}L + F_{KK}K = 0) \end{aligned}$$

[3]Note that Δ_k does not depend on individual initial wealth level k.

This decomposition shows that the pecuniary externality emerges from two channels. We call Δ_{k1} the "insurance effect" and Δ_{k2} the "redistribution effect". If there is no labor income risk ($\sigma_e^2 = 0$), then we can easily see that $\Delta_{k1} = 0$. Thus, Δ_{k1} captures the pecuniary externality due to idiosyncratic risk. On the other hand, if there is no wealth-inequality ($k'(k) = K$) for all k, then we can see that $\Delta_{k2} = 0$. Thus, Δ_{k2} captures the pecuniary externality due to wealth inequality. These two sources of pecuniary externalities were already discussed in DHKR, but we can show the decomposition *in additive terms* by exploiting the CARA-normal specification, which will be more helpful for the analysis with endogenous human capital presented below.

Then, let us look at the sign of each component of Δ_k. First, the sign of the insurance effect is always negative ($\Delta_{k1} < 0$). This negative insurance effect becomes stronger as the variance of the shock σ_e^2 increases. On the other hand, the sign of the redistribution effect is always positive ($\Delta_{k2} > 0$), which can be seen from the following. First, notice that

$$\Delta_{k2} \quad = \quad F_{KK} K \beta exp(\frac{\psi^2}{2} F_L^2 \sigma_e^2) \int u'(wL + rk'(k)) \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk).$$

The sign of the integral term is negative because

$$\begin{aligned} & \int u'(wL + rk'(k)) \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk) \\ = & \int_{k'(k) \geq K} u'(wL + rk'(k)) \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk) + \int_{k'(k) < K} u'(wL + rk'(k)) \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk) \\ < & \ u'(wL + rK) \int_{k'(k) \geq K} \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk) + u'(wL + rK) \int_{k'(k) < K} \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk) \\ = & \ u'(wL + rK) \int \left[\frac{k'(k)}{K} - 1\right] \Gamma(dk) = 0, \end{aligned}$$

and the sign of the term premultiplying the integral is also negative. Thus, in sum, the sign of Δ_{k2} is positive.

We now discuss the intuition of the sign of Δ_k, turning first to Δ_{k1}. In this economy, idiosyncratic labor shock is the only shock. Thus, labor income is risky, but capital income is not. If the planner decreases the wage rate and increases the interest rate, the risky part of the household income is scaled down, improving the welfare of risk-averse households. By decreasing capital the planner can decrease the wage rate and increase the interest rate. Thus, Δ_{k1} is always negative.

Second, the intuition of the positive Δ_{k2} follows. In period 1, there is dispersion of capital due to different amounts of k' across households. Then, the welfare of the capital-poor household can be improved by decreasing the interest rate and increasing the wage rate. Because the marginal utility of consumption-poor households is relatively higher and capital-poor households tend to be consumption poor in this economy, the planner with a utilitarian social welfare function wants to increase capital so that he can decrease the interest rate.

We see that the two channels through which the pecuniary externalities emerge have opposite implications for the level of capital. Thus, the sign of Δ_k depends on which effect dominates. DHKR carry out three different calibrations, and conclude that in a calibration that generates a realistic wealth dispersion for the US, the redistribution effect dominates ($\Delta_k > 0$) and thus there is underaccumulation of capital in a competitive equilibrium, relative to the constrained efficient allocation.

From the above analysis, we see that the key mechanism leading to a positive Δ_k is capital inequality in period 1, which results from endogenous saving decisions of households. On the other hand, labor income inequality in period 1 is exogenous, since labor income is determined by the shock. This endogenous dispersion of capital and fixed labor income inequality in the two-period model reflects the key mechanism behind the DHKR's quantitative result: for the earning process that generates highly dispersed wealth in the steady state, capital in a competitive equilibrium is underaccumulated relative to the constrained efficient allocation. In the next section, we investigate whether this underaccmulation result crucially depends on the exogenous labor income and endogenous capital dispersion.

2.2 Introduction of endogenous human capital

We observed that endogenous capital dispersion and exogenous labor income risk result in DHKR's quantitative result — underaccumulation of capital in a highly wealth-dispersed economy. In the real world, not only wealth but also labor productivity is endogenously dispersed due to human capital accumulation.[4] The natural question that then arises is whether DHKR's conclusion is overturned by endogenizing human capital. Thus, in this section, we incorporate human capital accumulation into the model and analyze the over or underaccumulation of capital in this economy.

We now describe an economy with endogenous human capital accumulation, which will be used in the remainder of this section. A household is born with initial wealth k, human

[4] We can also endogenize labor income by introducing labor choice. However, labor choice only affects labor income within period, whereas the human capital accumulation generates labor productivity dispersion in the succeeding periods in a multi-period model.

capital h, and learning ability A. In period 0, a household with (k, h, A) will decide how much to consume c_0, invest in physical capital k' and invest in human capital x_h.[5] At period 1, human capital $h' = g(A, h, x_h)$ is produced using the human capital production technology g, which is increasing in all three arguments and differentiable. Additionally, we assume that g is concave in h ($g_{hh} < 0$) and that there is complementarity between h and x_h ($g_{x_h h}$>0). There are two types of idiosyncratic risk to the human capital return, multiplicative risk η, and additive risk e. For notational simplicity, we denote that $s = (e, \eta)$. Both shocks are normally distributed as follows:

$$\begin{aligned} \eta &\sim iid \quad N(1, \sigma_\eta^2), \\ e &\sim iid \quad N(0, \sigma_e^2). \end{aligned}$$

Then, the budget constraints of the household with initial (k, h, A) and realized shock (e, η) will be

$$\begin{aligned} &c_0(k, h, A) + k'(k, h, A) + x_h(k, h, A) = k \\ &c_1(k, h, A, e, \eta) = w[\eta g(A, h, x_h(k, h, A)) + e] + rk'(k, h, A). \end{aligned}$$

The mean and variance of the period 1 consumption in this economy are:

$$E_s[c_1] = wg(A, h, x_h) + rk', \quad Var_s(c_1) = w^2[\sigma_\eta^2 g(A, h, x_h) + \sigma_e^2]$$

In this subsection and the next (2.2 and 2.3), we abstract away from the multiplicative shock to directly compare the analysis with that of the DHKR economy. Then, we analyze the effects of the multiplicative shock in section 2.4.

In this subsection, we also assume that there is no heterogeneity in (h, A) in order to focus on the pure effect of endogenizing human capital — the effect of simply making human capital a choice variable, abstracting from human capital dispersion. This assumption is relaxed in the next subsection.

2.2.1 Competitive Equilibrium

The household's problem for given initial k is as follows.

$$V(k) = \max_{x_h, k'} -\frac{1}{\psi} \exp\Big(-\psi(k - k' - x_h)\Big) - \beta \frac{1}{\psi} \exp\left(-\psi \left\{ wg(A, h, x_h) + rk' - \frac{\psi}{2} w^2 \sigma_e^2 \right\}\right)$$

[5] In a two-period model, we use this money investment model, but the main result is robust to the time investment model.

For simplicity, we do not impose any lower bound condition on k' and x_h. The first order conditions with respect to k' and x_h are as follows.

$$u'(c_0(k)) = \beta r E[u'(c_1)|k] \tag{3}$$

$$u'(c_0(k)) = \beta w g_{x_h}(A, h, x_h) E[u'(c_1)|k] \tag{4}$$

By dividing (3) by (4), we obtain the following no arbitrage condition.

$$r = w g_{x_h} \tag{5}$$

Notice that r is the marginal return to physical capital investment k', and wg_{x_h} is the marginal return to human capital investment x_h. From (5), we can see that the human capital investment $x_h(k)$ does not depend on k, which implies $h'(k) = g(h, A, x_h(k)) = H$, which does not depend on k. Since there is no heterogeneity in (h, A) in this section, every household has the same human capital in period 1.

Combining the Euler equation and the budget constraint, we obtain the following closed form solution for capital investment

$$k'(k) = \frac{1}{1+r}\left[\frac{1}{\psi} log(\beta r) + \frac{\psi}{2} w^2 \sigma_e^2 - wH + k - x_h\right], \quad \forall k.$$

Thus, we can see that $\frac{\partial k'(k)}{\partial k} = \frac{1}{1+r} > 0$, which generates a dispersion of wealth in period one.

The competitive firms have the same production function $F(K, H)$ as in the baseline economy. The only difference from the baseline economy is that the aggregate human capital H is no longer exogenous in this endogenous human capital economy. It is $H = g(h, A, x_h)$, which depends on endogenous human capital investment.

Next lemma shows that there exist initial human capital, learning ability, and distribution of initial wealth such that a competitive equilibrium in the endogenous human capital economy is identical to that in the DHKR economy.

Lemma 2. *The coefficient of risk aversion, discount factor, and variance of shocks are held constant across economies. Then, there exist a pair of initial human capital and learning ability (h, A) and a distribution of initial wealth $\Gamma(k)$ such that the competitive equilibrium of the endogenous human capital economy is identical to that of the DHKR economy.*

Proof We denote human capital investment which satisfies the following equation $\frac{r^{DHKR}}{w^{DHKR}} = g_{x_h}(h, A, x_h)$ by $X_h(r^{DHKR}, w^{DHKR}, h, A)$. Then, there exists a pair of (h, A) that satisfies

$L^{DHKR} = g(h, A, X_h(r^{DHKR}, w^{DHKR}, h, A))$. Then, for given (h, A), we can construct a distribution of wealth so that it satisfies $\Gamma(k) = \Gamma^{DHKR}(k^{DHKR} - X_h(r^{DHKR}, w^{DHKR}, h, A))$. Then, we can easily see that the allocations in the two economies are identical. Thus, the prices in the two economies are also identical. ■

With identical competitive equilibrium in both economies, we now analyze the constrained planner's problem in the endogenous human capital economy and check whether it has different implication on the constrained efficient allocation.

2.2.2 Constrained Efficient Allocation

The constrained social planner's problem in this endogenous human capital economy amounts to choosing the saving and human capital investment of each household with initial wealth k.

$$\max_{k'(k), x_h(k)} \int \begin{bmatrix} -\frac{1}{\psi}\exp\left(-\psi(k - k' - x_h)\right) \\ -\beta\frac{1}{\psi}\exp\left(-\psi\left\{F_H g(h, A, x_h) + F_K k' - \frac{\psi}{2}F_H^2\sigma_e^2\right\}\right) \end{bmatrix} \Gamma(dk)$$

The first order conditions with respect to $k'(k)$, $x_h(k)$ are as follows.

$$\begin{aligned} k'(\hat{k}): \quad & u'(c_0(\hat{k})) = \beta F_K E[u'(c_1)|\hat{k}] + \Delta_k && (6) \\ & \textit{where} \quad \Delta_k = \beta \int E[u'(c_1)|k]\left[F_{HK}H + F_{KK}k'(k) - \psi F_H F_{HK}\sigma_e^2\right]\Gamma(dk) \\ x'(\hat{k}): \quad & u'(c_0(\hat{k})) = \beta F_H g_{x_h} E[u'(c_1)|\hat{k}] + g_{x_h}\Delta_h && (7) \\ & \textit{where} \quad \Delta_h = \beta \int E[u'(c_1)|k]\left[F_{HH}H + F_{KH}k'(k) - \psi F_H F_{HH}\sigma_e^2\right]\Gamma(dk) \end{aligned}$$

From the first order condition with respect to capital, we can see that there is an additional term Δ_k due to the pecuniary externality of capital, as in the baseline economy. In addition, the first order condition with respect to human capital investment also has an additional term Δ_h due to a pecuniary externality of human capital. This implies that from the perspective of the constrained planner, not only the level of capital but also the level of human capital is inefficient. Moreover, there is a clear relationship between Δ_k and Δ_h. Using $F_{KH}K + F_{HH}H = 0$ and $F_{HK}H + F_{KK}K = 0$, we obtain the following relationship between Δ_k and Δ_h.

Proposition 3.

$$\Delta_h = -\frac{K}{H}\Delta_k$$

Thus, the sign of Δ_h is exactly the opposite of the sign of Δ_k. That is, if there is an additional marginal benefit to saving from the perspective of the constrained planner, then there is an additional marginal cost to investing in human capital. Thus, we can only focus on the sign of Δ_k. We also remark that, from now on, the variable of interest is the capital-labor ratio (K/H) not the level of capital (K) itself, because labor (H) is also endogenous variable and it is difficult to make definite statements about absolute levels when both capital (K) and labor (H) change.

We now analyze our main interest: the effect of introducing human capital on the optimal level of the capital-labor ratio. Does the endogeneity of human capital change the evaluation of whether there is overaccumulation of capital in a competitive equilibrium? In the next proposition, we answer this question in two parts. In the first part, we analyze whether an agent's welfare at a competitive equilibrium can be improved by reducing or increasing a small amount of aggregate capital, which is a local analysis in the neighborhood of a competitive equilibrium. In the second part, we analyze whether the constrained planner's optimal capital-labor ratio is higher than that of competitive equilibrium, which is a global analysis. If the constrained planner's problem is globally concave in the saving and human capital investment of every household, then the global analysis will be in accord with local analysis.

Before stating the proposition, we establish notation for local analysis. Recall that Δ_k is evaluated at a constrained efficient allocation, which is the marginal benefit of changing prices by increasing saving *at the constrained efficient allocation*. We now define Δ_k^{CE} which evaluates the marginal benefit of changing prices by increasing saving *at a competitive equilibrium*:

$$\Delta_k|_{CE} = \beta \int E\left[u'(c_1^{CE}|k)\right][F_{HK}H^{CE} + F_{KK}k'^{CE}(k) - \psi F_H F_{HK}\sigma_e^2]\Gamma(dk),$$

where F_{HK}, F_{KK}, and F_H are evaluated at $K^{CE} = \int k'^{CE}(k)\Gamma(dk)$ and $H^{CE} = g(A, h, x_h^{CE})$.

We can easily notice that Δ_k and Δ_k^{CE} exhibit the same equation but are evaluated at different allocations (at the constrained efficient allocation and competitive equilibrium). $\Delta_k|_{CE}$ is essentially the welfare effect of marginally increasing capital accumulation $k'(k)$ of every household at the competitive equilibrium[6], and the sign of Δ_k^{CE} will be the same as the

[6] Let's denote the welfare at an allocation $((k'(k))_k, x_h)$ by $W\left((k'(k))_k, x_h\right)$. Then

$$\begin{aligned}\int \frac{d}{dk'(k)} W\left(\left(k'^{CE}(k)\right)_k, x_h^{CE}\right)\Gamma(dk) &= \int \frac{dW}{dk'(k)}\Big|_{(r^{CE}, w^{CE})}\Gamma(dk) + \Delta_k|_{CE} \\ &= \Delta_k|_{CE},\end{aligned}$$

sign of Δ_k if the planner's problem is globally concave. With this notation, we now present the proposition.

Proposition 4. *Suppose that the competitive equilibria in both endogenous human capital economy and DHKR economy are identical. Then,*

(i) $sign(\Delta_k|_{CE}) = sign(\Delta_k^{DHKR}|_{CE})$

(ii) If the constrained planner's problems are globally concave in $k'(k)$ *and* x_h *in both economies, then* $sign(\Delta_k) = sign(\Delta_k^{DHKR})$.

Proof

(i) The competitive equilibrium allocations of the two economies are equivalent. Thus, $\Delta_k|_{CE} = \Delta_k^{DHKR}|_{CE}$.

(ii) Because the constrained planner's problem in the endogenous human capital economy is globally concave in $k'(k)$ and x_h, we know that if $\frac{dU}{dK}\Big|_{CE} = \Delta_k|_{CE} > 0$, then the capital allocation of the constrained planner's problem is higher than that of competitive equilibrium, which implies $\Delta_k > 0$. Similarly, if $\Delta_k^{DHKR}|_{CE} > 0$, then $\Delta_k^{DHKR} > 0$. Since $sign(\Delta_k|_{CE}) = sign(\Delta_k^{DHKR}|_{CE})$ by (i), we obtain $sign(\Delta_k) = sign(\Delta_k^{DHKR})$. ∎

Proposition 4 shows that the evaluation on the overaccumulation of capital does not change by simply endogenizing human capital. The intuition is as follows. Regardless of the introduction of human capital, the only way the planner can improve welfare is by changing prices — by either increasing r and decreasing w or increasing w and decreasing r. Moreover, by assuming no initial heterogeneity in (h, A), there is no endogenous dispersion of human capital in this two-period model, and by Lemma 2, there exist (h, A) and $\Gamma(k)$ such that the competitive equilibria in the economies with human capital and without human capital are equivalent. Thus, the planner can improve welfare in both economies that have the same competitive equilibrium allocations by changing the prices in the same way (either increasing r and decreasing w or vice versa). The only difference is that, in the DHKR economy, this price correction can only be achieved by changing capital, while in an endogenous human capital economy, the price correction is obtained by changing both capital and human capital (which is seen in the presence of Δ_h). Thus, the constrained planner's optimal level of

where the second equality follows from the optimality condition of household problem for given prices.

capital will be different in both economies, but the direction of change from the competitive equilibrium will be the same. In sum, endogenizing human capital itself does not overturn the under or overaccumulation result of DHKR.

2.3 Initial heterogeneity in human capital

In the previous section, we showed that endogenizing human capital itself does not overturn the conclusion of DHKR. The reason why we obtain the same conclusion is that even though we endogenize human capital, there is no endogenous human capital dispersion when there is no initial heterogeneity in human capital and learning ability, (h, A) in this two-period model. Recall that in section 2.2, we made this assumption — no heterogeneity in (h, A) — to focus on the effect of endogenizing human capital abstracting from human capital dispersion. However, in an infinite horizon, there will be endogenous human capital dispersion due to the shocks to human capital and productivity of human capital investment, which depends on the human capital stock. As we assumed heterogeneity in initial wealth k in order to capture endogenous wealth dispersion, we now introduce heterogeneity in initial human capital h, so that we can capture the endogenous human capital dispersion of the infinite horizon model in a two-period model, while obtaining an explicit analytical insight. Heterogeneity in the initial learning ability A will also help endogenous human capital dispersion, but for simplicity, we abstract from this feature in this section.

2.3.1 Competitive Equilibrium

The first order conditions of the household problem are equal to those of section 2.2 — (3) and (4) except that consumption is now expressed as function of (k, h), because there is also heterogeneity in h. From these first order conditions, we obtain the no-arbitrage condition $r - wg_{x_h} = 0$. By applying the implicit function theorem to this no-arbitrage condition, we can easily show that $\frac{\partial x_h}{\partial h} > 0$.[7] We then obtain $\frac{\partial h'}{\partial h} = g_{x_h} \frac{\partial x_h}{\partial h} > 0$. Thus, initial inequality in h generates human capital inequality in period 1. By the same argument as above, we also obtain $\frac{\partial k'}{\partial k} = \frac{1}{1+r} > 0$, which results in the endogenous dispersion of capital in period 1.

[7] We can write the no-arbitrage condition, $G(x_h) = r - wg_{x_h} = 0$. Then, from the implicit function theorem,

$$\frac{\partial x_h}{\partial h} = -\frac{\frac{\partial G}{\partial h}}{\frac{\partial G}{\partial x_h}} = -\frac{wg_{x_h h}}{wg_{x_h x_h}} > 0$$

2.3.2 Constrained Efficient Allocation

The constrained social planner's problem is as follows.

$$\max_{k'(k,h),x_h(k,h)} \int_{k,h} \left[\begin{array}{l} -\frac{1}{\psi}\exp\left(-\psi(k-k'-x_h)\right) \\ -\beta\frac{1}{\psi}\exp\left(-\psi\left\{F_H g(h,A,x_h)+F_K k'-\frac{\psi}{2}F_H^2\sigma_e^2\right\}\right) \end{array} \right]\Gamma(dk,dh)$$

The first order conditions with respect to $k'(\hat{k},\hat{h})$ and $x_h(\hat{k},\hat{h})$ are equal to those of section 2.2 — (6) and (7) except that consumption is now a function of $(\hat{k},\hat{h})$, individual human capital $h'(k,h)$ is not equal to aggregate human capital H, and Δ_k is integration over k and h:

$$\Delta_k = \beta\int_{k,h} E[u'(c_1)|k,h]\left[F_{HK}h'(k,h)+F_{KK}k'(k,h)-\psi F_H F_{HK}\sigma_e^2\right]\Gamma(dk,dh).$$

We can also easily notice that proposition 3 applies to this economy, implying that the sign of Δ_h is always the opposite of the sign of Δ_k. Thus, we can focus on the analysis of the sign of Δ_k only in order to evaluate whether the capital-labor ratio is too high in a competitive equilibrium compared to that of a constrained planner's problem.

In order to analyze the effects of the introduction of heterogeneity in human capital, we decompose Δ_k into insurance channel Δ_{k1} and redistribution channel Δ_{k2} ($\Delta_k=\Delta_{k1}+\Delta_{k2}$).

$$\begin{aligned}
\Delta_{k1} &= -\psi F_H F_{HK}\sigma_e^2\beta\int_{k,h} E[u'(c_1)|k,h]\Gamma(dk,dh)\\
\Delta_{k2} &= \beta\int_{k,h} E[u'(c_1)|k,h]\left[F_{HK}h'(k,h)+F_{KK}k'(k,h)\right]\Gamma(dk,dh)\\
&= \underbrace{F_{KK}K\beta\int_{k,h}E[u'(c_1)|k,h]\left[\frac{k'(k,h)}{K}-1\right]}_{\Delta_{k2,K}} + \underbrace{F_{KK}K\beta\int_{k,h}E[u'(c_1)|k,h]\left[1-\frac{h'(k,h)}{H}\right]}_{\Delta_{k2,H}}
\end{aligned}$$

The introduction of heterogeneity in initial human capital h does not have direct impact on the insurance channel. Of course, the introduction of heterogeneity could change the value of Δ_{k1} indirectly due to a change in consumption inequality. Regardless of the change in magnitude, however, the insurance effect is always negative.

On the other hand, introducing heterogeneity in h has a direct qualitative impact on the redistribution channel. We can see that the redistribution channel Δ_{k2} has an additional term $\Delta_{k2,H}$ relative to that in the absence of heterogeneity in human capital. In the absence of heterogeneity in h, there is no human capital inequality in period 1. Thus, there is only a redistribution effect due to capital heterogeneity ($\Delta_{k2,K}$). In the presence of heterogeneity in

h, however, there is also a redistribution effect due to human capital heterogeneity ($\Delta_{k2,H}$). Then, how does the additional term affect the sign of the redistribution channel? As we showed in section 2.1.2, in the absence of heterogeneity in h, the sign of the redistribution channel is always positive. However, in the presence of heterogeneity in h, even the redistribution channel could be negative, since the sign of additional term $\Delta_{K2,H}$ is negative due to exactly the opposite reason why $\Delta_{k2,K}$ is positive.

Recall that in the case of capital inequality, the welfare of the capital-poor is improved when the interest rate ($r = F_K$) decreases, that is, when K/H increases. Because the marginal utility of the consumption poor is relatively high, if the consumption poor tend to be capital poor, the planner will want to increase K/H, which is the intuition of a positive $\Delta_{k2,K}$. On the other hand, in the case of human capital inequality, the welfare of the human capital poor is improved when the wage rate ($w = F_H$) decreases, that is when K/H decreases. Since the marginal utility of the consumption poor is high, if the consumption-poor tend to be human capital poor, the planner will want to decrease K, which is the intuition for negative $\Delta_{K2,H}$. Then, depending on which redistribution channel dominates, the redistribution channel can be either positive or negative.

The relative inequality of capital and human capital and their correlation with consumption inequality determine the sign of Δ_{k2}, which can be seen by rewriting Δ_{k2} in the following way.

$$\Delta_{k2} = F_{KK} K \beta \left\{ cov\left(E[u'(c_1)|k,h], \frac{k'(k,h)}{K} \right) - cov\left(E[u'(c_1)|k,h], \frac{h'(k,h)}{H} \right) \right\}$$

From this equation, we can directly obtain the next proposition.

Proposition 5. *If* $cov\left(E[u'(c_1)|k,h], \frac{k'(k,h)}{K}\right) < cov\left(E[u'(c_1)|k,h], \frac{h'(k,h)}{H}\right)$, *then* $\Delta_{k2} < 0$, *thus* $\Delta_k < 0$.

This proposition shows that the relative size of the correlation between consumption inequality and wealth inequality vs. the correlation between consumption inequality and human capital inequality is the key determinant of the sign of Δ_k. We can also notice that because consumption inequality itself is determined by wealth inequality and human capital inequality, the sign of Δ_k depends on the relative size of wealth inequality to human capital inequality and the correlation between wealth inequality and human capital inequality. In section 3, we analyze how these features affect the sign of Δ_k more explicitly.

2.4 Risky return to human capital investment.

We have so far analyzed the constrained efficient allocation when there is only additive shock to human capital investment to make our analysis directly comparable to that of DHKR. Most studies in the human capital literature, however, model human capital formation with multiplicative shock (for example, Huggett et al. (2011), Guvenen et al. (2009)) to capture the properties of risk on human capital return, and this modeling with multiplicative shock is consistent with empirical finding that the variation of the human capital return increases in education.[8] Since it is likely that increased risk renders the insurance channel of pecuniary externality stronger, we analyze how the multiplicative shock changes the constraint allocation.

We now assume that there are two types of idiosyncratic risk, multiplicative risk η, and additive risk e, to compare the role of multiplicative shocks to that of additive shocks.[9] Multiplicative risk makes an additional human capital investment increase not only the mean return to human capital but also its variance, while with additive risk only, human capital investment only affects the mean return to human capital.

Recall that with multiplicative shock, the variance of period-1 consumption is expressed as $Var_s(c_1) = w^2[\sigma_\eta^2 g(A, h, x_h)^2 + \sigma_e^2]$, showing that it depends on both the variance of the multiplicative shock (σ_η^2) and the human capital investment (x_h). Thus, the consumption risk is endogenous in this economy.

Then, the first order condition of the constrained planner's problem with respect to $k'(k, h)$ has the same form as (6) with different Δ_k:

$$\Delta_k = \beta \int_{k,h} E_s[u'(c_1)|k, h] \begin{bmatrix} F_{HK} h'(k, h) + F_{KK} k'(k, h) \\ -\psi F_H F_{HK} \{\sigma_\eta^2 h'(k, h)^2 + \sigma_e^2\} \end{bmatrix} \Gamma(dk, dh),$$

where Δ_k is additively decomposed of following two terms.

$$\Delta_{k1} = -\beta\psi F_H F_{HK} \int_{k,h} E_s[u'(c_1)|k, h]\{\sigma_\eta^2 h'(k, h)^2 + \sigma_e^2\}\Gamma(dk, dh)$$

$$\Delta_{k2} = F_{KK} K\beta \int_{k,h} E_s[u'(c_1)|k, h] \left[\frac{k'(k, h)}{K} - 1\right] \Gamma(dk, dh)$$
$$+F_{KK} K\beta \int_{k,h} E_s[u'(c_1)|k, h] \left[1 - \frac{h'(k, h)}{H}\right] \Gamma(dk, dh)$$

[8]For example, table 1 of Palacios-Huerta (2003) shows that human capital investment increases not only the mean but also the variance of the return to human capital in the US, during 1964 – 1996.

[9]Singh (2010) calls these shocks to human capital specialization risk (multiplicative risk) and career risk (additive risk).

We can notice direct effect of introducing multiplicative shock on Δ_{k1} — an additional negative term in Δ_{k1}, $-\beta\psi F_H F_{HK} \int_{k,h} E_s[u'(c_1)|k,h]\sigma_\eta^2 h'(k,h)^2$. This direct effect stems from the fact that in the presence of the multiplicative shock, the investment of human capital increases risk. Because of the increased risk, the insurance effect strengthens. The indirect effects on Δ_{k1} come from the response of a household to the increased risk. The household decreases the human capital to reduce the exposure to risk, but increases capital investment due to increased precautionary incentive. This response to risk will change the distribution of consumption in period 1, which has ambiguous effects on Δ_{k1}. There are also indirect effects on $\Delta_{k,2}$ through change of distribution whose directions are ambiguous.

In sum, if the direct insurance effect of introducing multiplicative shock dominates other indirect effects, the introduction of multiplicative shock will decrease the optimal level of capital relative to labor in the constrained efficiency.

2.5 Implementation of the constrained efficient allocation

In this section, we briefly discuss how we can implement the constrained efficient allocation using taxes, to see the implication of pecuniary externalities on optimal tax system. Of course, the constrained efficient allocation can be attained if the government directly modifies each consumer's savings and human capital investments. But, we can also implement the constrained efficient allocation as a competitive equilibrium using a tax(subsidy)-transfer(lump-sum tax) system which is history dependent and induces no reallocation of income across individuals or realization of idiosyncratic shocks.

In the two-period model, history-dependent taxes and transfers are represented by the initial heterogeneity-dependent taxes and transfers. The tax(subsidy)-transfer(lump-sum tax) system is then completely characterized by the linear proportional tax rates of capital income and labor income $(\tau_k(k,h), \tau_l(k,h))$ and transfers $(T(k,h,s))$, which are functions of initial wealth and human capital (k,h) and idiosyncratic shock realization $s = (\eta, e)$.

Proposition 6. *There exists a triple of history-dependent capital income tax rates, labor income tax rates, and income transfer $(\tau_k(k,h), \tau_l(k,h), T(k,h,s))$ which implements the constrained planner's allocation.*

Proof By setting capital income tax rates and labor income tax rates in the following way, the Euler equation of the competitive equilibrium and the constrained planner's problem become equivalent:

$$\tau_k(k,h) = -\frac{\Delta_k}{\beta(F_k - \delta)E_s[u'(c_1)|k,h]},$$

and $\tau_l(k,h)$ is the solution of the following equation:

$$\psi f_H \sigma_\eta^2 h' \tau_l(k,h)^2 + (1 - 2\psi f_H \sigma_\eta^2 h')\tau_l(k,h) = -\frac{\Delta_h}{\beta f_H E_s[u'(c_1)|k,h]}, \quad \forall (k,h)$$

Then, the following transfer function which imposes no income transfer across agents implements the constrained efficient allocation.

$$T(k,h,s) = \tau_k r k'(k,h) + \tau_l w[\eta h'(k,h) + e]$$

■

Notice that if there is no multiplicative shock ($\sigma_\eta^2 = 0$), labor income tax rates are set to $\tau_l(k,h) = -\frac{\Delta_h}{\beta f_H E[u'(c_1)|k,h]}$. With this tax system, the underaccumulation of capital implies subsidy to the capital income and taxes on labor (human capital) income. Subsidy to capital income is regressive (tax rate is increasing in capital income), but tax on human capital income is progressive (tax rate is decreasing in human capital return). But, we want to emphasize that this feature of tax system is the implication of the pure pecuniary externalities of incomplete market and the tax-transfer system implementing this constrained efficient allocation is very information-demanding, which requires all of the history of shocks of each individual.[10] If there is some restrictions on the available policy tools, then there can be additional insurance/redistribution effects arising from tax system itself, then optimal tax should also consider those effects.

3 Effects of correlation and relative concentration

In section 2.3.2, we could see that the relative size of redistribution channel through wealth dispersion to that through human capital dispersion depends on the cross-correlation between wealth and human capital and relative inequalities. In this section, we analyze the effect of increasing the cross-correlation and increasing earnings inequality on the constrained efficient allocation, focusing on their impact on Δ_k — the additional marginal benefit of saving in the Euler equation of the constrained efficient allocation.

[10]In a two-period model, this tax-transfer is not that much information-demanding, but it will be quite complicated in a multi-period model.

Cross correlation and inequalities are endogenous objects which are determined by combination of many primitives. Then, why do we focus on the effect of these endogenous statistics? First, it is not easy to analyze the effect of more fundamental factors (such as primitive parameters or modeling choices) on the constrained efficient allocation. If we know how correlation and inequalities affect the constrained efficient allocation, however, then it can be helpful to conjecture the effect of primitives/model choices on the constrained efficient allocation by predicting how those factors change inequalities and cross correlation. Second, inequalities and cross correlation are empirically measurable and available statistics.[11] Then, using these statistics, we might predict how the marginal tax reform from the current tax system can enhance welfare by considering pecuniary externalities.

In this section, we first investigate the effects of cross correlation and inequalities on the constrained efficient allocation analytically and numerically, and then we briefly discuss how this result can be used to conjecture the effect of primitives/model choices on the constrained efficient allocation.

3.1 Effects of correlation and inequalities on the constrained efficiency

In order to analyze the effect of the correlation and inequalities, we directly impose assumptions on the distribution of wealth and human capital. For the theoretical analysis, we assume simple joint uniform distribution of wealth and human capital and analytically show the effect of increasing the correlation and human capital inequalities. Then we provide a simple numerical example to check quantitative significance of the correlation between wealth and human capital on the optimal capital-labor ratio. It shows that the two economies with the same inequalities in wealth and human capital, but with different correlation coefficient of wealth and human capital, can end up with different conclusions on whether the capital-labor ratio is too high.

3.1.1 Analytical results with uniform wealth-human capital distribution

We maintain the two-period CARA-normal specification and abstract from multiplicative risk for simplicity. In order to analyze the effect of human capital inequalities and

[11] Statistics for the inequalities can be obtained easily for most of the countries but the correlation between wealth and earnings is available for fewer countries.

the correlation of capital and human capital explicitly, we make an assumption directly on the period-1 distribution of capital and human capital in a competitive equilibrium. For theoretical analysis, we assume that joint distribution of capital k' and human capital h' at a competitive equilibrium has the bivariate uniform distribution. With this tractable distribution assumption, we can analytically show how the correlation between capital and human capital and the relative earnings-Gini to wealth-Gini affect the evaluation, whether the capital-labor ratio is too high in a competitive equilibrium.

We focus on the local analysis in the neighborhood of a competitive equilibrium. That is, we consider how the relative inequalities and correlation affect $\Delta_k|_{CE}$ — the marginal benefit of changing prices by increasing saving at a competitive equilibrium (defined in section 2.2.2). The reason we focus on the local analysis at a competitive equilibrium is that we are interested in how the correlation and inequalities in human capital without government policies changes the implication of the pecuniary externalities, and the local welfare effect of increasing a small amount of capital, $\Delta_k|_{CE}$ is the one that can be expressed in terms of those statistics at a competitive equilibrium. On the other hand, in a global analysis, the term Δ_k in the constrained planner's Euler equation is expressed in terms of statistics of the constrained efficient allocation, which is more involved and does not give direct interpretation. Thus, we focus on $\Delta_k|_{CE}$. As in Proposition 4, if the constrained planner's problem is globally concave in saving and human capital investment, then the sign of $\Delta_k|_{CE}$ will directly imply the sign of global analysis (the sign of Δ_k) : if a small increase (decrease) in capital at the competitive equilibrium improves welfare, then the efficient capital-labor ratio will be higher (lower) than that of the competitive equilibrium.

We can start the analysis by rewriting $\Delta_k|_{CE}$ as the integration over the distribution of period 1-capital and period 1-human capital in a competitive equilibrium.

$$\begin{aligned}
\Delta_k|_{CE} &= \beta \int_{k,h} E\left[u'\left(\begin{array}{c} w(h'^{CE}(k,h)+e) \\ +rk'^{CE}(k,h) \end{array}\right)\Big|k,h\right]\left[\begin{array}{c} -\psi F_H F_{HK}\sigma_e^2 + F_{KK}K \\ \times\left\{\frac{k'^{CE}(k,h)}{K} - \frac{h'^{CE}(h,k)}{H}\right\} \end{array}\right]\Gamma(dk,dh) \\
&= \beta \int_{k',h'} E\left[u'\left(\begin{array}{c} w(h'+e) \\ +rk' \end{array}\right)\Big|k',h'\right]\left[\begin{array}{c} -\psi F_H F_{HK}\sigma_e^2 \\ +F_{KK}K\left\{\frac{k'}{K} - \frac{h'}{H}\right\} \end{array}\right]\Gamma_1(dk',dh'), \qquad (8)
\end{aligned}$$

where, $\Gamma_1(k',h')$ is the joint distribution of period-1 wealth (k') and human capital (h') in a competitive equilibrium. The period-1 joint density of $\gamma_1(k',h')$ in a competitive equilibrium is given by

$$\gamma_1(k',h') = \begin{cases} \frac{1}{2}\rho & \text{if } (k',h') = (k'_1,h'_1) \\ \frac{1}{2}(1-\rho) & \text{if } (k',h') = (k'_1,h'_2) \\ \frac{1}{2}(1-\rho) & \text{if } (k',h') = (k'_2,h'_1) \\ \frac{1}{2}\rho & \text{if } (k',h') = (k'_2,h'_2), \end{cases} \tag{9}$$

where $k'_1 = (1-\theta_k)K$, $k'_2 = (1+\theta_k)K$, $h'_1 = (1-\theta_h)H$, $h'_2 = (1+\theta_h)H$, $\theta_k, \theta_h > 0$.

Then, the marginal distributions of k' and h' are discrete uniform with probabilities $pr(k'_1) = pr(k'_2) = \frac{1}{2}$, and $pr(h'_1) = pr(h'_2) = \frac{1}{2}$, respectively. With this simple form of distribution, we can generate various combinations of inequalities and correlation, because the wealth Gini and human capital Gini computed from this distribution are $\frac{\theta_k}{2}$ and $\frac{\theta_h}{2}$, respectively and the correlation between k' and h' is $2\rho - 1$. Thus, by varying θ_k, θ_h, and ρ, we can get joint distribution with various wealth-inequality, human capital-inequality, and correlation between wealth and human capital.

With this simple distribution assumption, we can show the effect of correlation and inequality on $\Delta_k|_{CE}$ analytically.

First, we analyze the effect of increasing correlation on $\Delta_k|_{CE}$ for fixed wealth inequality and earnings inequality. That is, we analyze the effect of increasing ρ on $\Delta_k|_{CE}$ for fixed θ_k and θ_h. The next proposition presents the effect of increasing correlation on insurance channel ($\Delta_{k1}|_{CE}$) and redistribution channel ($\Delta_{k2}|_{CE}$).

Proposition 7. *Suppose that the distribution of period-1 wealth and human capital (k', h') is given by (9), and we assume $F(K,H) = K^\alpha H^{1-\alpha}$. Then,*

(i) $\frac{\partial \Delta_{k1}|_{CE}}{\partial \rho} < 0$

(ii) If $\theta_k \geq \theta_h$ and $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, then $\frac{\partial \Delta_{k2}|_{CE}}{\partial \rho} > 0$

Proof See the Appendix ■

The first part of Proposition 7 shows that for fixed wealth-inequality and human capital-inequality, an increase in the correlation between wealth and human capital strengthens the insurance channel (more negative insurance channel). The second part of Proposition 7 shows that if the wealth inequality is higher than the human capital inequality and their gap is not too large, then the increase in the correlation between wealth and human capital will strengthen the redistribution channel through wealth dispersion. The conditions of *(ii)* are relevant, because in the real world data, wealth is usually more concentrated than human capital and the second condition — $\frac{\theta_k}{\theta_h} = \frac{K-Gini}{H-Gini} \leq \frac{1-\alpha}{\alpha} \approxeq 2$ — usually holds.[12]

The intuition for the second part of the proposition 7 is the following. To improve welfare through the redistribution channel, income decomposition of the consumption poor is the key, because the consumption poor have relatively higher marginal utility. Since the wealth inequality is greater than human capital inequality, the consumption poor are likely to be relatively poorer in wealth than in human capital. Thus, the redistribution channel through wealth dispersion will dominate the redistribution channel through human capital dispersion, implying that increasing K/H-ratio is welfare improving. As the correlation coefficient between wealth and human capital (ρ) increases, the wealth poor tend to be also human capital poor which reduces their consumption even more, leading to higher marginal utility and thus reinforcing the redistribution channel through wealth dispersion. Conversely, as the correlation (ρ) decreases, the wealth poor is less likely to be consumption poor, weakening the redistribution channel through wealth dispersion.[13]

An increase in correlation will drive insurance and redistribution channels in opposite directions, the numerical example below (in section 3.1.2) will show that it is likely that its impact on redistribution channel will be stronger when wealth inequality relative to human capital inequality is high enough.

Next, we analyze the effect of increasing human capital inequality on $\Delta_k|_{CE}$ for a fixed wealth inequality and correlation. That is, we analyze the effect of increasing θ_h for fixed θ_k and ρ. The next proposition presents the effect of increasing human capital inequality on the insurance and redistribution channels.

Proposition 8. *Suppose that the distribution of period-1 wealth and human capital* (k', h') *is given by (9), and the production function has the functional form:* $F(K, H) = K^\alpha H^{1-\alpha}$. *Assume that* $\frac{\theta_k}{\theta_h} < \frac{1-\alpha}{\alpha}$. *Then,*

(i) $\frac{\partial \Delta_{k1}|_{CE}}{\partial \theta_h} < 0$

[12]The conditions of (ii) are only sufficient, not necessary.

[13]DHKR also briefly mention the possibility of stronger redistribution channel for higher correlation, when discussing the effect of serially correlated earnings shock (on page 2440). In the DHKR economy with exogenous labor shock, the result of Proposition 7 also holds. In fact, an increase of the cross correlation is more likely to increase the Δ_{k2} in the exogenous income shock case, because in human capital model, there is some counter effect. Increasing ρ reduces the degree of poverty in wealth relative to that in human capital (captured by the increase of $(\frac{k'}{K} - \frac{h'}{H})$ for the consumption-poor (in equation (8)) which weakens the relative redistribution channel through wealth inequality somewhat, while this effect is absent for the exogenous shock case.

(ii) If $\frac{(1-\rho)(\theta_k+\theta_h)}{\rho(\theta_k-\theta_h)} > \frac{u(wh_2'+rk_2')+u(wh_1'+rk_1')}{u(wh_1'+rk_2')+u(wh_2'+rk_1')}$, *then* $\frac{\partial \Delta_{k2}|_{CE}}{\partial \theta_h} < 0$.

Proof See the Appendix ■

Proposition 8 shows that for fixed correlation and wealth inequality, an increase in human capital inequality decreases $\Delta_k|_{CE}$, under the conditions which ensure that the gap between the wealth inequality and human capital inequality is not too large. As we mentioned above, $\frac{\theta_k}{\theta_h} < \frac{1-\alpha}{\alpha}$ usually holds in the real world. We can also see that the condition of (ii) is easily satisfied, if the gap of wealth-inequality and human capital inequality $(\theta_k - \theta_h)$ is not too big. Thus, for empirically plausible parameterization, an increase in human capital-inequality have implication toward overaccumulation of capital.

In sum, a low correlation between capital and human capital and a high human capital-inequality are likely to place the planner's optimal capital-labor ratio below that of the competitive equilibrium. In the next subsection, we investigate the quantitative significance of the correlation on the optimal capital-labor ratio.

3.1.2 Numerical example

Next, we provide a numerical example to check the quantitative significance of correlation and show that two economies with different correlation coefficients can have opposite conclusions on whether the capital-labor ratio is too high in a competitive equilibrium, even though the two economies have exactly the same inequalities in wealth and human capital. Ideally, we want to calibrate the model to different combinations of wealth inequality, human capital inequality, and correlation between wealth and human capital to see the effect of each component, but it is not easy to match various targets since these objects are endogenously determined. Then, it is difficult to decompose the pure effect of increasing the correlation. Thus, in this section, we simply check the quantitative significance of the correlation by directly assuming that the endogenous period-1 distribution of wealth and human capital is bivariate lognormal distribution, so that we can capture various inequalities and correlations abstractly.

The economic environment of this section is the same as that in section 2.2, except for the distribution of wealth and human capital. In this section, period-1 wealth and human capital (k', h') have a bivariate lognormal distribution, which is given by

$$\left(log(\frac{k'}{K}), log(\frac{h'}{H})\right) \sim N\left(\left(\begin{array}{c}\mu_{lk}\\ \mu_{lh}\end{array}\right), \left(\begin{array}{cc}\Sigma_{kk} & \Sigma_{kh}\\ \Sigma_{hk} & \Sigma_{hh}\end{array}\right)\right),$$

where $K = E[k']$ ($\mu_{lk} = -0.5\Sigma_{kk}$) and $H = E[h']$ ($\mu_{lh} = -0.5\Sigma_{hh}$).

The correlation coefficient of wealth and earning is computed as follows:

$$corr(k', h' + e) = corr(k', h') = \frac{exp(\Sigma_{kh}) - 1}{\sqrt{exp(\Sigma_{kk}) - 1}\sqrt{exp(\Sigma_{hh}) - 1}}.$$

Thus, for fixed Σ_{kk} and Σ_{hh}, the correlation coefficient is determined by Σ_{kh}, and by varying Σ_{kk}, Σ_{hh}, and Σ_{kh}, we can match various combinations of the wealth-Gini, human-capital-Gini and correlation coefficient between wealth and earnings.[14]

We now numerically investigate the effect of increasing the correlation coefficient between wealth and earnings for fixed wealth-Gini and earnings-Gini. By changing the parameters of (assumed) log-normal distribution (period-1 distribution), we show the possible quantitative significance of the correlation without considering how this correlation is endogenously generated. However, we want to understand this simple numerical example in connection with the infinite horizon model that we study in section 4. For that, we think of the period-1 of this numerical example as the last period of a long-horizon model — that is, the model period of this numerical example is one year — and set the parameters to mimic some features of the US economy. We set the capital-income share α to $\frac{1}{3}$ ($F(K, H) = K^{\alpha}H^{1-\alpha}$), and choose $\beta = 0.96$ and $\delta_k = 0.08$. The mean of human capital is normalized to 1 and the mean of capital is chosen so that the annual interest rate is 4 percent. We fix $\sigma_e^2 = 0.1$, and choose Σ_{kk} and Σ_{hh} so that wealth-Gini and earnings-Gini are 0.8 and 0.6, respectively.

In Figure 1, we display $\Delta_k|_{CE}$ by the correlation coefficient for three different values of the risk aversion parameter, ψ (1, 1.5, 2). We can see that as the correlation coefficient increases, $\Delta_k|_{CE}$ increases. More importantly, Figure 1 shows that $\Delta_k|_{CE}$ changes the sign from negative to positive as the correlation coefficient increases, which implies that two economies with the same wealth-Gini and human capital-Gini can have the opposite evaluations on whether capital-labor ratio is too high, depending on the size of the correlation. Figure 2 displays the decomposition of $\Delta_k|_{CE}$ into insurance channel ($\Delta_{k1}|_{CE}$) and redistribution channel ($\Delta_k|_{CE}$) for $\psi = 1.5$, and shows that the effect of correlation on $\Delta_k|_{CE}$ is driven by a stronger redistribution channel effect, as we discussed in Proposition 7 with the simple uniform distribution.

[14] The wealth-Gini of this lognormal distribution is $2\Phi(\sqrt{\Sigma_{kk}}/\sqrt{2}) - 1$, where Φ is the cumulative distribution function of the standard normal. Thus, wealth-Gini is determined by Σ_{kk}. On the other hand, the earnings-Gini of this economy depends on both the variance of e-shock (σ_e^2) and the variance of relative human capital ($var(\frac{h'}{H}) = exp(\Sigma_{hh}) - 1$)), because earnings of this economy are defined by $w(h' + e)$.

Figure 1: Effects of correlation on $\Delta_k|_{CE}$

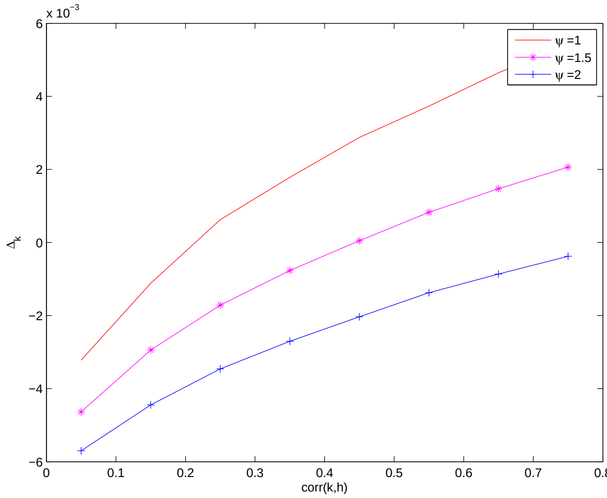
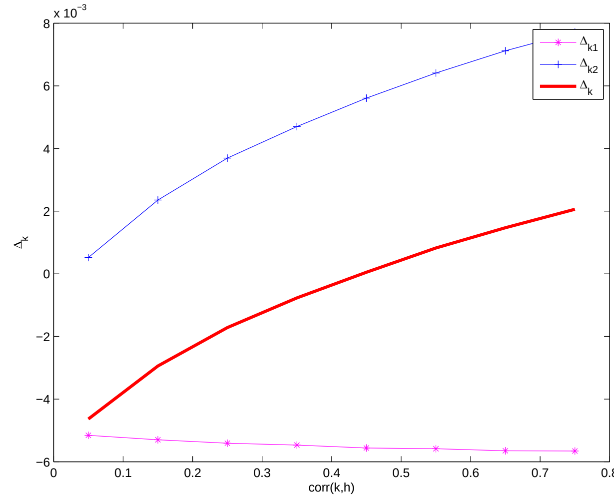


Figure 2: Decomposition of $\Delta_k|_{CE}$ ($\psi = 1.5$)



This numerical example might have some implications in the real world. In the US, the correlation coefficient between wealth and earnings has been significantly increasing. Díaz-Giménez et al. (1997) report that the correlation coefficient in 1992 measured by Survey of Consumer Finances (SCF) was 0.23, and Díaz-Giménez et al. (2011) report that the correlation coefficient in 2007 measured by SCF was 0.48.[15] If we take the above numerical example, the observation that the sign of $\Delta_k|_{CE}$ based on the 1992 statistics is negative (implying overaccumulation of capital), while the sign of $\Delta_k|_{CE}$ based on 2007 statistics is positive (implying underaccumulation of capital) might imply that the evaluation on the overaccumulation of capital in a competitive equilibrium has been changed in the US. This numerical example also sheds some light on the cross-country analysis. Countries that have very low correlation coefficients, such as Sweden and Spain, are likely to exhibit overaccumulation of capital in a competitive equilibrium.

3.2 Application of the correlation/inequality analysis

So far, we have analyzed how the endogenous correlation and inequalities affect the constrained efficient allocation. We now discuss that this analysis might be used to predict the effect of more fundamental factors that determine these endogenous statistics such as primitive parameters and the modeling choices on the constrained efficient allocation.

[15] Even though there has also been some increase of wealth inequality and earnings inequality (the wealth-Gini increased from 0.78 to 0.82 and the earnings-Gini increased from 0.63 to 0.74), these increase are less significant than the increase in correlation.

Variance of ability or variance of shock Let's first discuss the effects of increasing the variance of ability or variance of shock on the constrained efficient allocation. If the variance of ability increases, then it is likely to increase the human capital inequality relative to the wealth inequality which leads to the stronger redistribution effect through human capital dispersion, but its impact on the cross correlation is ambiguous. If the redistribution effect through higher human capital inequality is strong, then we can conjecture that the increase of the variance of ability is likely to change the implication toward the overaccumulation of capital in a competitive equilibrium.

On the other hand, increasing the variance of the shock have more ambiguous implications. There will be stronger insurance effect (implication toward overaccumulation of capital), but the redistribution effect is ambiguous. This is because we might expect increase in wealth inequality due to relative less investment in riskier human capital but higher variance of shock and reduced investment in human capital can have opposite effects on human capital inequality.

Effects of persistent shock If the persistence of human capital shock increases, the the correlation between the wealth and human capital is likely to increase which leads to the implication toward underaccumulation of capital.

Effects of money investment Vs. time investment Next, we discuss the effect of modeling choice about the human capital formation — money investment versus time investment. This human capital modeling choice can have impact on the constrained efficiency by changing the correlation between the wealth and human capital. We discuss the effects in the presence of iid shock first and then with persistent shock.

In the presence of iid shock, the cross correlation is likely to be lower in the time investment model compared to the money investment model, because the cost of time investment depends on shock while the return to human capital investment does not depend on the current shock. On the other hand, cost and benefit of the money investment does not depend on the current shock if the shock is iid. Thus, the wealth-rich who has experienced sequence of good shocks is likely to have invested less on human capital in the time investment model, implying that the cross correlation might be lower in the time investment model.

In the presence of persistent shock, the cross correlation is likely to be lower again in the time investment model, but because of different reason. Now, both cost and return of time investment depend on the current shock while only return of the money investment model depends on the current shock. Thus, the wealth-rich who has experienced sequence of good shocks is likely to have invested more on human capital in the money investment case, implying higher correlation in the money investment model.

Thus, time investment model is more likely to imply overaccumulation of capital than the money investment model for the same inequalities of wealth and human capital, although the mechanism that leads to this result is different depending on the persistence of shock.

4 Infinite horizon model

We now study the infinite horizon version of the endogenous risky human capital model. Using a two-period model, we observed that endogenizing human capital dispersion and risky human capital accumulation may lead to the evaluation that the capital-labor ratio in a competitive equilibrium is too high. In this section, we quantitatively examine whether the capital-labor ratio is too high in a risky human capital model from the constrained planner's perspective, using the infinite horizon model calibrated to actual data (wealth-Gini and earnings-Gini). DHKR show that in a standard incomplete market economy with exogenous labor dispersion (shock) which generates realistic wealth dispersion, capital accumulation in a competitive equilibrium is lower than that of constrained efficient allocation. We are interested in whether introducing risky human capital accumulation, which generates endogenous human capital dispersion, overturns DHKR's conclusion on the optimal capital-labor ratio.

In this economy, the preferences of infinitely lived households are represented by expected lifetime utility,

$$E_0 \left\{ \sum_t \beta^t u(c_t) \right\},$$

where the period utility u has the constant relative risk aversion (CRRA) form, $\frac{c^{1-\sigma}}{1-\sigma}$. Introducing human capital accumulation into the model, we closely follow Huggett et al. (2011). Household with invariant learning ability A that has human capital h_t in period t and invest time[16] s_t will have human capital h_{t+1} at period $t+1$

$$h_{t+1} = \eta_{t+1} g(h_t, s_t, A),$$

[16]This time investment for human capital production is slightly different from money investment in the two-period model of section 2, but the main result we got in the theoretical analysis does not change. We adopt this time investment model for infinite horizon quantitative investigation because this is the model of Huggett et al. (2011), and we use their estimates. In addition, it is much easier to guarantee stationarity in an infinite horizon with this time investment model than with the money investment model.

where η is an idiosyncratic (multiplicative) shock to human capital. The human capital production technology is given by (see e.g., Ben-Porath (1967)):

$$g(h, s, A) = (1 - \delta_h)h + A(sh)^{\phi}.$$

The only source of risk the household faces, η is i.i.d. across agents and time, and is drawn from the log-normal distributions every period,

$$\log \eta \sim N(\mu_\eta, \sigma_\eta^2),$$

where $E[\eta] = 1$. There is initial heterogeneity in learning ability. At period 0, A is drawn from a log-normal distribution:

$$\log A \sim N(\mu_A, \sigma_A^2).$$

The period budget constraint of the household is

$$c_t + k_{t+1} = k_t(1 + r) + w(1 - s_t)h_t.$$

As the budget constraint shows, households can insure against idiosyncratic shock either by accumulating capital or human capital. We also notice that earnings at period t are $w(1 - s_t)h_t$, where h_t is human capital at period t and $1 - s_t$ is available time allocated to market work.

Competitive firms have production function $f(K_t, L_t)$, where K_t is aggregate capital and L_t is aggregate labor. We assume that f is a Cobb-Douglas function $f(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$. The depreciation rate for capital is denoted by δ_k. Due to competitive firms, wages (rental price of human capital) and interest rates (rental price of capital) are determined by the marginal productivity of labor and the marginal productivity of capital, respectively:

$$w = f_L(K, L), \quad r = f_K(K, L) - \delta_k.$$

We now state household problem recursively. The state variables of the individual household are (k, h, A), and $(k, h, A) \in S = \mathbb{K} \times \mathbb{H} \times \mathbb{A}$. The probability measure Φ is defined over the Borel sets of S. As in DHKR, agents have to know aggregate state variable, Φ in order to compute prices. The law of motion of the distribution is written as $\Phi' = F(\Phi)$.

Then, the household's problem is,

$$\begin{aligned} v(\Phi, k, h; A) &= \max_{c,k',s} u(c) + \beta E_{\eta'} v(\Phi', k', \eta' g(h, s, A); A) \\ s.t. \quad & c + k' = k[1 + r(\Phi)] + (1 - s)hw(\Phi) \quad and \quad \Phi' = F(\Phi) \end{aligned}$$

The solution of the household's problem is $c(\Phi, k, h; A)$, $k'(\Phi, k, h; A)$, $s(\Phi, k, h; A)$. The intertemporal conditions the solution satisfies are

$$\begin{aligned} u'(c) &\geq \beta(1 + r(\Phi'))E_{\eta', e'}\left[u'(c')\right] \\ w(\Phi)hu'(c) &\leq \beta g_s(h, s, A)w(\Phi')E_{\eta'}\left[u'(c')\eta'\left\{1 - s + \frac{g_h(h', s', A)h'}{g_s(h', s', A)}\right\}\right], \end{aligned}$$

where the equality hold when $k' \geq 0$ and $s \leq 1$, respectively.[17]

Using the decision rules $k'(\Phi, k, h; A)$, $s(\Phi, k, h; A)$, transition function is computed as follows.

$$Q(\Phi, k, h, A, B; k', s) = \int_{\eta'} f(\eta')\chi_{k'(\Phi,k,h;A)\in B_k}\chi_{\eta' s(\Phi,k,h;A)\in B_h}\chi_{A\in B_A} d\eta',$$

where $f(\eta')$ is log normal density of η', and χ is indicator function. Given this transition function, we define the updating operator $T(\Phi, Q)$ as

$$\Phi'(B) = T(\Phi, Q)(B) = \int_S Q(\Phi, k, h, A, B; k', s)d\Phi.$$

Then, we can define a recursive competitive equilibrium.

Definition 9. *A recursive competitive equilibrium is value function, $v(\phi, k, h; A)$, policy functions $c(\Phi, k, h; A)$, $k'(\Phi, k, h; A)$, $s(\Phi, k, h; A)$, prices $r(\Phi)$, $w(\Phi)$, and aggregate law of motion, F s.t.*

1. *Given w, r, and F, policy functions c, k', s solve the household's problem and associated value function is v.*

2. *The price functions satisfy $r(\Phi) = f_K(K, L)$, and $w(\Phi) = f_L(K, L)$, where $K = \int kd\Phi$ and $L = \int(1 - s(\Phi, k, h; A))hd\Phi$.*

3. *$F(\Phi) = T(\Phi, Q)$*

A steady state for this economy is an invariant distribution $\tilde{\Phi}$, which is fixed point of the updating operator T.[18]

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[17] $s \geq 0$ does not bind, since $\lim_{s\to 0} g_s(h, s, a) = \infty$

[18] Even though the production technology is constant returns to scale in human capital which is accumulated over time, there is a steady state because, at the individual level, household faces a diminishing marginal product of human capital ($\phi < 1$).

4.1 Constrained Efficient Allocation

We now characterize the constrained efficient allocation in an infinite horizon economy. As in the two-period economy, the constrained social planner cannot complete the market using transfers, but internalize the price effects. We assume a utilitarian planner, who is maximizing ex-ante lifetime utility. The planner's problem is then defined as follows:

$$\Omega(\Phi) = \max_{\substack{k'(k,h;A),\\ s(k,h;A)}} \int_S u\Big(k(1+r(\Phi)) + (1-s(k,h;A))hw(\Phi) - k'(k,h;A)\Big)\, d\Phi + \beta\Omega(\Phi') \quad (10)$$

$$s.t. \quad \Phi' = T(\Phi, Q(\cdot;k',s)).$$

We denote the optimal decision rule by $k'^*(\Phi,k,h;A)$ and $s^*(\Phi,k,h;A)$. The following proposition provides the first order conditions of the constrained planner's problem, which is analogous the ones we analyzed in a two period model.

Proposition 10. *If the distribution Φ admits a density, the first order necessary conditions of problem (10) are the following functional equation in the decision rule k'^*, s^*: For all $\{k,h,A\} \in S$,*

1. *First order condition with respect to k' is:*

$$\begin{aligned}
&u'\Big(k[1+r(\Phi)] + (1-s^*(\Phi,k,h;A))hw(\Phi) - k'^*(\Phi,k,h;A)\Big) \\
&\quad \geq \beta[1+r(\Phi')]E_{\eta'}\Big[u'\Big(c^*(\Phi,\Phi',k,h,\eta';A)\Big)\Big] + \Delta_k, \quad where \\
&c^*(\Phi,\Phi',k,h,\eta';A) = k'^*(\Phi,k,h;A)[1+r(\Phi')] \\
&\quad +(1-s^*(\Phi',k'^*(\Phi,k,h;A),\eta' g(h,s^*(\Phi,k,h;A),A),A);A)\eta' g(s^*(\Phi,k,h;A),h,A)w(\Phi') \\
&\quad -k'^*(\Phi',k'^*(\Phi,k,h;A),\eta' g(h,s^*(\Phi,k,h;A),A);A), \\
&\Delta_k = \beta\int_S u'\Big(k'[1+r(\Phi')] + (1-s^*(\Phi',k',h';A))h'w(\Phi) - k'\Big) \\
&\qquad\qquad \times \Big[k' f_{KK}(K',L') - (1-s^*(\Phi',k',h';A))h' f_{LK}(K',L')\Big]\, d\Phi',
\end{aligned}$$

   where the inequality becomes equality if $k'^*(\Phi,k,h;A) > 0$.

2. *First order condition with respect to s is:*

$$\begin{aligned}
&w(\Phi)hu'\Big(k[1+r(\Phi)]) + (1-s^*(\Phi,k,h;A))hw(\Phi) - k'^*(\Phi,k,h;A)\Big) \\
&\leq \beta w(\Phi')g_s(h,s^*(\Phi,k,h;A),A) \\
&\times E_{\eta'}\left[u'(c^*(\Phi,\Phi',k,h,\eta';A))\eta'\left\{\begin{array}{l}(1-s^*(\Phi',k'^*(\Phi,k,h;A),\eta' g(h,s^*(\Phi,k,h;A),A);A)) \\ +\frac{g_h(\eta' g(h,s^*(\Phi,k,h;A),A),s^*(\Phi',k'^*(\Phi,k,h;A),\eta' g(h,s^*(\Phi,k,h;A),A);A),A)}{g_s((\eta' g(h,s^*(\Phi,k,h;A),A),s^*(\Phi',k'^*(\Phi,k,h;A),\eta' g(h,s^*(\Phi,k,h;A),A);A),A)} \\ \times\eta' g(h,s^*(\Phi,k,h;A),A)\end{array}\right\}\right] \\
&+(1-s^*(\Phi,k,h;A))g_s(h,s^*(\Phi,k,h;A),A)\Delta_h,
\end{aligned}$$

where the inequality becomes equality if $s^*(\Phi, k, h; A) < 1$.

3. $\Delta_k = \beta \int_S \int_{\eta'} \tilde{\Delta}_k(\Phi, \Phi', k, h, \eta'; A) d\eta' d\Phi,$
$\Delta_h = -\frac{K}{L}\beta \int_S \int_{\eta'} \eta' \tilde{\Delta}_k(\Phi, \Phi', k, h, \eta'; A) d\eta' d\Phi,$ *where*
$\tilde{\Delta}_k(\Phi, \Phi', k, h, \eta'; A) = u'(c^*(\Phi, \Phi', k, h, \eta'; A))$
$\times \left[k'^*(\Phi, k, h; A) f_{KK}(K', L')(1 - s^*(\Phi', k', h'; A)\eta' g(h, s^*(\Phi, k, h; A), A) f_{LK}(K', L')\right]$

Proof See the appendix. ■

From this proposition, we see that the Euler equation of the constrained efficient allocation in an infinite horizon model is the dynamic generalization of the formula we obtained in the two-period model, (6) and (7). Thus, the sign of extra term Δ_k in the Euler equation determines whether the capital-labor ratio in a competitive equilibrium is too high.

For the purpose of quantitatively investigating over/underaccumulation, we might want to simply compare the ratio $\frac{K}{H}$ between the equilibrium and efficient allocation. Then, why do we show the first order condition in Proposition 10? From these first order conditions, we can see that the over/underaccumulation in this infinite horizon model is indeed determined by the magnitude of the insurance channel and the two redistribution channels of the pecuniary externalities that we analyzed in section 2 with the two-period model. The Euler equations also show how the increase (decrease) in saving and human capital compared to a competitive equilibrium is distributed among the rich and poor. If Δ_k is positive, the constrained optimum allocates even more saving to the rich, because marginal utility of consumption of the rich is small, while the extra marginal benefit of saving Δ_k is the same for everyone. Similarly, we see that if Δ_h is positive, the constrained planner allocates even more human capital investment to the rich.

In the next section, we quantitatively investigate the optimal capital-labor ratio. We investigate this focusing on the steady state, as in DHKR. It is important, however, to notice that this steady state analysis is not the result of maximizing steady-state welfare, but the result of maximizing ex-ante life time welfare including the transition to the steady state.

4.2 Quantitative analysis

We now quantitatively examine whether the capital-labor ratio in a competitive equilibrium is too high compared to that of the constrained optimum, when the model is calibrated

Table 1: Calibrated parameter values

Category	Symbol	Parameter Value
Preference	$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$	$\sigma = 2$
	β	$\beta = 0.9376$
Technology	$Y = K^\alpha L^{1-\alpha}$	$\alpha = 0.36$
	δ_k	$\delta_k = 0.08$
Human capital	$h' = \eta g(h, s, A)$	$\phi = 0.75$
	$g(h, s, A) = (1 - \delta_h)h + A(hs)^\phi$	$\delta_h = 0.04$
Human capital shock	$\log \eta \sim N(\mu_\eta, \sigma_\eta^2)$	$\sigma_\eta = 0.111$
Initial heterogeneity	$\log A \sim N(\mu_A, \sigma_A^2)$	$(\mu_A, \sigma_A) = (-2.336, 0.222)$

to generate realistic dispersion of wealth and human capital. Using this realistic calibration, we show that despite the introduction of risky human capital accumulation that generates endogenous dispersion of human capital, capital-labor ratio in a competitive equilibrium is lower than that of constrained optimum, even though the magnitude of capital-labor ratio in a constrained optimum is smaller than that of DHKR.

The calibrated parameters are summarized in Table 1. The parameters for preference and technology are standard. Relative risk aversion σ is 2, and discount factor β is calibrated to place the equilibrium interest rate at 4 percent, as the calibration of DHKR (originally Díaz et al. (2003)) does. Capital share α of the Cobb-Douglas production function ($K^\alpha L^{1-\alpha}$) is equal to 0.36 and the depreciation rate of capital is set to 0.08, which are standard and exactly equal to that of DHKR.

The return to scale, ϕ of the human capital production function is set to 0.75, and the depreciation rate of human capital δ_h is set to 0.04, which are in the middle of estimates in the literature.[19]

The parameters of the human capital shock process are (μ_η, σ_η), and the parameters of the distribution of initial learning ability are (μ_A, σ_A). The standard deviation of shock to human capital σ_η and the standard deviation of initial learning ability σ_A are important parameters that determine the earnings dispersion. σ_η is set to 0.111, which is the estimate of the Huggett et al. (2011), and the initial learning ability is calibrated to 0.222, which generates an earnings Gini index of 0.60 — the target of calibration of DHKR(originally

[19] Browning et al. (1999) surveyed the estimates of these parameter. Table 2.3 of Browning et al. (1999) shows that the estimates of ϕ lie in the range 0.5 to over 0.9, and the estimates of δ_h lie in the range 0 to 0.089. Huggett et al. (2011) uses $\phi = 0.7$ and $\delta_h = 0$.

Díaz et al. (2003)). μ_η is set to guarantee $E[\eta] = 1$, and μ_A is normalized to set the steady state output without shock (first best output) to 1.

We first comment on some features of the competitive equilibrium. The competitive equilibrium of this economy has large precautionary savings. As a result, the aggregate capital-labor ratio is 1.37 times as large as that of an economy without shocks (first best). Additionally, this economy can generate a Gini index of wealth of 0.783, which is close to the U.S data. Another important feature of competitive equilibrium is that the correlation of wealth and human capital in this economy is 0.706, which is higher than the 0.48 in the US data[20] — which is the the 2007 correlation coefficient measured by SCF and reported in Díaz-Giménez et al. (2011). Based on our analysis in section 3, this higher correlation of the model is likely to imply a higher constrained efficient capital ratio, and this implies that there can be some bias on the evaluation of the capital-labor ratio in a competitive equilibrium toward the underaccumulation of capital.

The capital-labor ratio of the constrained efficient allocation is higher than that of the competitive equilibrium in the long run, maintaining DHKR's conclusion — underaccumulation of capital. The capital-labor ratio of the constrained optimum is 3.60 times as large as that of the first best (without shock) and 2.63 times larger than that of the competitive equilibrium. Thus, the capital-labor ratio in the competitive equilibrium is too low relative to the capital-labor ratio in the constrained efficient allocation, implying underaccumulation of capital. In DHKR's (section 5.2) economy with exogenous labor income shock, the capital-labor ratio of a constrained efficiency is 8.5 times larger than that of the deterministic economy and 3.65 times larger than that of the competitive equilibrium. This implies that the degree to which the capital-labor ratio in a competitive equilibrium is lower than the optimal ratio in an economy with endogenous human capital is about 70 percent of that in an economy with exogenous labor income shock.

Another notable finding is that inequalities in both wealth and human capital of the constrained efficient allocation are lower than those of a competitive equilibrium. Earnings-Gini of the constrained efficient allocation gets lower because negative Δ_h has stronger negative impact on the human capital accumulation of households with lower marginal utility (higher consumption) who tend to have higher earnings. With this observation, one might conjecture that stronger impact of positive Δ_k on wealth accumulation of relatively rich household would lead to higher wealth inequality. However, decrease of interest rate lowers wealth inequality, and this interest rate effect dominates the inequality-generating channel of positive

[20]We tried various calibrations. However, it turns out that, it is difficult to generate low correlation between wealth and human capital while matching wealth inequality and human capital inequality with this standard human capital model.

Table 2: Steady state for the infinite horizon economy

	First Best	Competitive Equilibrium	Constrained Optimum
Capital-labor ratio	4.070	5.565	14.634
Capital-output ratio	2.465	3.000	5.569
Interest rate	6.660%	4.000%	-1.536%
wealth Gini	0.0	0.783	0.738
CV[a] of wealth	0.0	2.605	1.807
earning Gini	0.0	0.600	0.402

[a]coefficient of variation

Δ_k because decreased consumption inequality resulting from decreased earnings inequality weakens the inequality-generating channel. This feature is different from constrained efficiency of DHKR whose wealth inequality is almost equal to that of a competitive equilibrium. That is, in an economy with endogenous human capital dispersion, the constrained efficiency improve ex-ante welfare by changing prices so that it can decrease both wealth inequality and human capital inequality ex-post.

5 Conclusion

We have analyzed whether the pecuniary externalities of the incomplete market imply that the capital-labor ratio is too high in a competitive equilibrium with endogenous human capital dispersion. Analytically, we showed that introducing endogenous human capital dispersion and human capital risk into a standard incomplete market economy with exogenous shock can overturn the implication of pecuniary externalities from under accumulation of capital to too high capital-labor ratio. Higher human capital inequality and lower correlation of wealth and human capital tend to have the constrained planner's evaluation that a competitive equilibrium exhibit too high capital-labor ratio. A quantitative investigation using calibration that generates realistic wealth inequality and human capital inequality, however, shows that even if we introduce risky human capital accumulation, the capital-labor ratio of a competitive equilibrium is lower than that of a constrained efficient allocation in the long run. Thus, the conclusion of DHKR is not overturned, but the magnitude of this ratio in an economy with human capital is smaller than that of a standard incomplete market economy with exogenous labor.

As we discussed in our quantitative analysis, the correlation of wealth and human capital of the model economy, which is higher than that found in actual US data, could be the reason behind the underaccumulation of capital. Thus, developing a model that generates low correlation while maintaining high wealth inequality and earnings inequality and investigating the optimum level of capital-labor ratio might be called for to get more precise quantitative evaluation. Also, further analysis on how different types of human capital model can change the implication of pecuniary externalities might be an interesting future research.

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Appendix

A Proof of proposition 7

We first rewrite the $\Delta_{k1}|_{CE}$ and $\Delta_{k2}|_{CE}$, using some simple algebra.

$$\begin{aligned}
\Delta_{k1}|_{CE} &= -\psi F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \int_{k',h'} u'(wh'+rk')\Gamma_1(dk',dh') \\
&= -\psi F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \begin{bmatrix} \frac{\rho}{2} u'(wh_1'+rk_1') + \frac{1-\rho}{2} u'(wh_2'+rk_1') \\ \frac{1-\rho}{2} u'(wh_1'+rk_2') + \frac{\rho}{2} u'(wh_2'+rk_2') \end{bmatrix}, \\
\Delta_{k2}|_{CE} &= F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \int_{k',h'} u'(wh'+rk') \left[\frac{k'}{K} - \frac{h'}{H}\right] \Gamma_1(dk',dh') \\
&= F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \begin{bmatrix} \frac{\rho}{2} u'(wh_1'+rk_1') \left[\frac{k_1'}{K} - \frac{h_1'}{H}\right] + \frac{1-\rho}{2} u'(wh_2'+rk_1') \left[\frac{k_1'}{K} - \frac{h_2'}{H}\right] \\ \frac{1-\rho}{2} u'(wh_1'+rk_2') \left[\frac{k_2'}{K} - \frac{h_1'}{H}\right] + \frac{\rho}{2} u'(wh_2'+rk_2') \left[\frac{k_2'}{K} - \frac{h_2'}{H}\right] \end{bmatrix}.
\end{aligned}$$

(i) By differentiating $\Delta_{k1}|_{CE}$ with respect to ρ, we get

$$\frac{\partial \Delta_{k1}|_{CE}}{\partial \rho} = -\frac{\psi}{2} F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \begin{bmatrix} (u'(wh_1'+rk_1') - u'(wh_2'+rk_1')) \\ -(u'(wh_1'+rk_2') - u'(wh_2'+rk_2')) \end{bmatrix} < 0,$$

where the last inequality holds because u' is decreasing ($u'' < 0$) and convex ($u''' > 0$).

(ii) By differentiating $\Delta_{k2}|_{CE}$ with respect to ρ, we get

$$\frac{\partial \Delta_{k2}|_{CE}}{\partial \rho} = \frac{1}{2} F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2) \begin{bmatrix} (\theta_k - \theta_h)(u'(wh_2'+rk_2') - u'(wh_1'+rk_1')) \\ -(\theta_k + \theta_h)(u'(wh_1'+rk_2') - u'(wh_2'+rk_1')) \end{bmatrix}.$$

Since $\theta_k \geq \theta_h$, $wh_2' + rk_2' > wh_1' + rk_1'$, the first term in the bracket is negative. If $\frac{\theta_k}{\theta_h} \leq \frac{1-\alpha}{\alpha}$, then $wh_1' + rk_2' < wh_2' + rk_1'$, by the following.

$$\begin{aligned}
&(wh_1'+rk_2') - (wh_2'+rk_1') \\
&= \left((1-\alpha)K^\alpha H^{1-\alpha}\frac{h_1'}{H} + \alpha K^\alpha H^{1-\alpha}\frac{k_2'}{K} - \delta_k k_2'\right) - \left((1-\alpha)K^\alpha H^{1-\alpha}\frac{h_2'}{H} + \alpha K^\alpha H^{1-\alpha}\frac{k_1'}{K} - \delta_k k_1'\right) \\
&= K^\alpha H^{1-\alpha}\left[((1-\alpha)(1-\theta_h) + \alpha(1-\theta_k)) - ((1-\alpha)(1+\theta_h) + \alpha(1-\theta_k))\right] - \delta_k(k_2' - k_1') \\
&< 2K^\alpha H^{1-\alpha}\left[\alpha\theta_k - (1-\alpha)\theta_h\right] \leq 0
\end{aligned}$$

Thus, $u'(wh_1'+rk_2') - u'(wh_2'+rk_1') > 0$, and the second term in the bracket is also negative. Since $F_{KK} < 0$ and the whole bracket is negative, $\frac{\partial \Delta_{k2}|_{CE}}{\partial \rho} > 0$

B Proof of proposition 8

(i) By differentiating $\Delta_{k1}|_{CE}$ with respect to θ_h, we get

$$\frac{\partial \Delta_{k1}|_{CE}}{\partial \theta_h} = -\frac{\psi}{2} F_H F_{HK} \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2)(1-\alpha) K^\alpha H^{1-\alpha}$$
$$\times \left[\ \rho \left(u''(wh_2' + rk_2') - u''(wh_1' + rk_1') \right) + (1-\rho) \left(u''(wh_2' + rk_1') - u''(wh_1' + rk_2') \right) \ \right]$$

Since u'' is increasing ($u''' > 0$), and $wh_2' + rk_2' > wh_1' + rk_1'$, the first term in the bracket is positive. If $\frac{\theta_k}{\theta_h} < \frac{1-\alpha}{\alpha}$, then $wh_2' + rk_1' > wh_1' + rk_2'$ as we showed in the proof of proposition 7. Thus, the second term in the bracket is also positive. Thus, we get $\frac{\partial \Delta_{k1}|_{CE}}{\partial \theta_h} < 0$.

(ii) By differentiating $\Delta_{k2}|_{CE}$ with respect to θ_h, we get

$$\frac{\partial \Delta_{k2}|_{CE}}{\partial \theta_h} = F_{KK} K \beta exp(\frac{\psi^2}{2} F_H^2 \sigma_e^2)$$
$$\times \begin{bmatrix} \frac{\rho}{2} \left(u'(wh_1' + rk_1') - u'(wh_2' + rk_2') \right) + \frac{1-\rho}{2} \left(u'(wh_1' + rk_2') - u'(wh_2' + rk_1') \right) \\ + \frac{\rho}{2}(\theta_k - \theta_h)(1-\alpha) K^\alpha H^{1-\alpha} \left(u''(wh_2' + rk_2') + u''(wh_1' + rk_1') \right) \\ - \frac{1-\rho}{2}(\theta_k + \theta_h)(1-\alpha) K^\alpha H^{1-\alpha} \left(u''(wh_1' + rk_2') + u''(wh_2' + rk_1') \right) \end{bmatrix}$$

The first and second terms in the bracket are positive. Thus, if the sum of third and fourth terms are positive, then $\frac{\partial \Delta_{k2}|_{CE}}{\partial \theta_h} < 0$. We can easily show that the sum of third and fourth terms are positive, if $\frac{(1-\rho)(\theta_k+\theta_h)}{\rho(\theta_k-\theta_h)} > \frac{u''(wh_2'+rk_2')+u''(wh_1'+rk_1')}{u(wh_1'+rk_2')''+u''(wh_2'+rk_1')} = \frac{u(wh_2'+rk_2')+u(wh_1'+rk_1')}{u(wh_1'+rk_2')+u(wh_2'+rk_1')}$, where the last equality holds because $u''(c) = \psi^2 u(c)$.

C Proof of proposition 10

Proof We prove the proposition based on a sequential formulation of the constrained-efficiency problem. We denote a density of the distribution of capital, human capital and learning ability at period t by ϕ_t. Then, given density ϕ_t, a sequence of the policy rule $\{k'{}_t(\phi_t, k, h; A), s_t(\phi_t, k, h; A)\}$ of the constrained planner's problem must solve

$$\begin{aligned} \max_{c_t, k_t', s_t} \quad & \sum_t \beta^t \int_S u(c_t) \phi_t(k, h, A) dS \\ s.t. \quad & c_t + k_t'(\phi_t, k, h; A) = k[1 + f_K(K(\phi_t), L(\phi_t))] + (1 - s_t(\phi_t, k, h; A)) h f_L(K(\phi_t), L(\phi_t)) \\ & h_{t+1} = \eta' g(h, s_t(\phi_t, k, h; A), A) \\ given \quad & \phi_1 \\ where \quad & K(\phi_t) = \int_S k \ \phi_t dS, \quad L(\phi_t) = \int_S (1 - s_t(\phi_t, k, h; A)) h \ \phi_t dS. \end{aligned}$$

Therefore, the planner's optimal policy rules k'^* and s^* that instructs a household with (k, h, A) today to save $k'^*(k, h; A)$ and invest on human capital $s^*(k, h; A)$ given density of today ϕ and k'' and h'' (thus, s') tomorrow must maximize

$$\begin{aligned} \max_{k',s} \quad & \int_S u\left(k[1 + f_K(K(\phi), L(\phi))] + (1 - s)hf_L(K(\phi), L(\phi)) - k'\right)\phi dS \\ & +\beta \int_S u\left(k'[1 + f_K(K'(\phi'), L'(\phi'))] + (1 - s')h'f_L(K'(\phi), L'(\phi)) - k''\right)\phi' dS \\ s.t. \quad & h' = \eta' g(h, s, A) \\ where \quad & \phi' \text{ is the density of distribution } \Phi' = T(\Phi, Q(\cdot; k', s)) \ . \end{aligned}$$

Using the change of variable, we can show that it is equivalent to maximizing

$$\max_{k',s} \quad \int_S \left[\begin{array}{l} u\left(k[1 + f_K(K(\phi), L(\phi))] + (1 - s)hf_L(K(\phi), L(\phi)) - k'\right) \\ +\beta \int_{\eta'} u\left(k'[1 + f_K(K'(\phi'), L'(\phi'))] + (1 - s')h'f_L(K'(\phi), L'(\phi)) - k''\right) f(\eta')d\eta' \end{array} \right] \phi dS.$$

We now derive the first order conditions using variational approach. First, we derive the first order condition with respect to k'. Variational policy rule,

$$k'^{\epsilon}(\phi, k, h; A) = k'^*(\phi, k, h; A) + \epsilon\chi_{h=h_0, A=A_0, k\geq k_0}$$

should be suboptimal. Define the welfare with variational policy by

$$\begin{aligned} \Psi_k(\epsilon) \quad = \quad & \int_S \Big[u\left(k[1 + f_K(K(\phi), L(\phi)] + (1 - s^*(\phi, k, h; A))hf_L(K(\phi), L(\phi) - k'^{\epsilon}(\phi, k, h, A)\right) \\ & +\beta \int_{\eta'} u \left(\begin{array}{l} k'^{\epsilon}(\phi, k, h; A)[1 + f_K(K'(T(\phi, k'^{\epsilon})), L'(\phi'))] \\ +(1 - s')\eta' g(h, s^*, A)f_L(K'(T(\phi, k'^{\epsilon})), L'(\phi')) - k'' \end{array} \right) f(\eta')d\eta'\Big]\phi dS. \end{aligned}$$

Then, derivative with respect to ϵ at 0 of $\Phi_k(\epsilon)$ must be 0:

$$\begin{aligned} \frac{d}{d\epsilon}\Psi_k(0) = & \int_{k_0}^{\infty} \left[\begin{array}{l} -u'\left(\begin{array}{l} k[1 + f_K(K(\phi), L(\phi))] + (1 - s^*(\phi, k, h_0; A_0)) \\ \times h_0 f_L(K(\phi), L(\phi)) - k'^*(\phi, k, h_0; A_0) \end{array} \right) \\ +\beta[1 + f_K(K'(\phi'), L'(\phi'))] \\ \quad \times \int_{\eta'} u' \left(\begin{array}{l} k'^*(\phi, k, h_0; A_0)[1 + f_K(K'(\phi'), L'(\phi'))] \\ +(1 - s')\eta' g(h, s^*, A)f_L(K'(\phi'), L'(\phi')) - k'' \end{array} \right) f(\eta')d\eta' \end{array} \right] \phi(k, h_0, A_0)dk \\ & +\beta \int_S \int_{\eta'} f(\eta') \left[\begin{array}{l} u' \left(\begin{array}{l} k'^*(\phi, k, h; A)[1 + f_K(K'(\phi'), L'(\phi'))] \\ +(1 - s')\eta' g(h, s^*, A)' f_L(K'(\phi), L'(\phi)) - k'' \end{array} \right) \\ \times \left\{ \begin{array}{l} k'^*(\phi, k, h; A)f_{KK}(K'(\phi'), L(\phi')) \\ +(1 - s')\eta' g(h, s^*, A)f_{LK}(K'(\phi'), L'(\phi')) \end{array} \right\} \int_{k_0}^{\infty} \phi(\tilde{k}, h_0, A_0)d\tilde{k} \end{array} \right] d\eta' \phi dS \\ = & \ 0. \end{aligned}$$

Since the right hand side of the above equation is constant function equal to 0. Thus, its derivative with respect to k_0 must be 0. That is, for all k_0, h_0, A_0,

$$
\begin{aligned}
& -u'\Big(k_0[1+f_K(K(\phi),L(\phi))]+(1-s^*(\phi,k_0,h_0;A_0))h_0 f_L(K(\phi),L(\phi))-k'^*(\phi,k_0,h_0;A_0)\Big) \\
+ \quad & \beta[1+f_K(K'(\phi'),L'(\phi'))]\int_{\eta'} u'\left(\begin{array}{c} k'^*(\phi,k_0,h_0;A_0)[1+f_K(K'(\phi'),L'(\phi'))] \\ +(1-s')\eta' g(h,s^*,A)f_L(K'(\phi'),L'(\phi'))-k'' \end{array}\right) f(\eta')d\eta' \\
+ \quad & \beta\int_S\int_{\eta'} f(\eta')\Big[u'\Big(k'[1+f_K(K'(\phi'),L'(\phi'))]+(1-s')\eta' g(h,s^*,A)f_L(K'(\phi),L'(\phi))-k''\Big) \\
& \times\Big\{k'^*(\phi,k,h;A)f_{KK}(K'(\phi'),L(\phi'))+\eta' g(h,s^*,A)f_{LK}(K'(\phi'),L'(\phi'))\Big\}\Big]d\eta'\phi dS=0
\end{aligned}
$$

That is, for all (k,h,A)

$$
\begin{aligned}
& u'\Big(k[1+r(\Phi)])+(1-s^*(\Phi,k,h;A))hw(\Phi)-k'^*(\Phi,k,h;A)\Big) \\
= \quad & \beta[1+r(\Phi')]E_{\eta'}\Big[u'\Big(c^*(\Phi,\Phi',k,h,\eta';A)\Big)\Big]+\Delta_k \\
& where \quad \Delta_k=\beta\int_S u'\Big(k'[1+r(\Phi')]+(1-s^*(\Phi',k',h';A))h'w(\Phi)-k'^*(\Phi',k',h';A)\Big) \\
& \qquad \times\Big[k' f_{KK}(K',L')-(1-s^*(\Phi',k',h';A))h'F_{LK}(K',L')\Big]d\Phi'
\end{aligned}
$$

Next, we derive the first order condition with respect to s. The variational policy rule

$$
s^\epsilon(\phi,k,h;A)=s^*(\phi,k,h;A)+\epsilon\chi_{k=k_0,A=A_0,h\geq h_0}
$$

should be suboptimal. Define

$$
\begin{aligned}
\Psi_s(\epsilon)=\int_S & \left[\begin{array}{l} u\Big(k[1+f_K(K(\phi),L(\phi)]+(1-s^\epsilon(\phi,k,h;A))hf_L(K(\phi),L(\phi)-k'^*(\phi,k,h,A)\Big) \\ +\beta\int_{\eta'} u\left(\begin{array}{c} k'^*(\phi,k,h,A)[1+f_K(K'(\phi'),L'(T(\phi,s^\epsilon)))] \\ +(1-s')\eta' g(h,s^\epsilon,A)f_L(K'(\phi'),L'(T(\phi,s^\epsilon)))-k'' \end{array}\right) f(\eta')d\eta' \end{array}\right]\phi dS, \\
& where \quad s'(s^\epsilon) \text{ is } s' \text{ s.t. } \eta'' g(\eta' g(s^\epsilon,h,A),s',A)=h''.
\end{aligned}
$$

Then, derivative with respect to ϵ at 0 of $\Phi_s(\epsilon)$ must be 0.

$$
\begin{aligned}
\frac{d}{d\epsilon}\Psi_s(0)= & \int_{h_0}^\infty\left[\begin{array}{l} -u'\left(\begin{array}{c} k_0[1+f_K(K(\phi),L(\phi))]+(1-s^*(\phi,k_0,h;A_0)) \\ \times hf_L(K(\phi),L(\phi))-k'^*(\phi,k_0,h;A_0) \end{array}\right) \\ +\beta g_s(h,s^*,A_0)f_L(K'(\phi'),L'(\phi'))\int_{\eta'} u'\Big(k'^*[1+f_K(K'(\phi'),L'(\phi'))] \\ +(1-s')\eta' g(h,s^*(\phi,k_0,h;A_0),A_0)f_L(K'(\phi'),L'(\phi'))-k''\Big) \\ \times\eta'\left\{1-s'+\frac{g_h(h',s',A_0)h'}{g_s(h',s',A_0)}\right\}f(\eta')d\eta' \end{array}\right]\phi(k_0,h,A_0)dh \\
& +\beta\int_S\int_{\eta'} f(\eta')\Big[u'\Big(k'[1+f_K(K'(\phi'),L'(\phi'))]+(1-s'))h' f_L(K'(\phi),H'(\phi))-k''\Big) \\
& \quad \times\Big\{k'^*(\phi,k,h;A)f_{KL}(K'(\phi'),L'(\phi'))+(1-s')\eta' g(h,s^*,A)f_{LL}(K'(\phi'),L(\phi'))\Big\} \\
& \quad \times\int_{h_0}^\infty\phi(k_0,\tilde{h},A_0)d\tilde{h}\Big]\eta'\phi dS=0
\end{aligned}
$$

Since the right hand side of the above equation is constant function equal to 0. Thus, its derivative with respect to h_0 must be 0. That is, for all k_0, h_0, A_0,

$$\begin{aligned}
& -u'\Big(k_0[1+f_K(K(\phi),L(\phi))]+(1-s^*(\phi,k_0,h_0;A_0))h_0 f_L(K(\phi),L(\phi))-k'^*(\phi,k_0,h_0;A_0)\Big) \\
+ \quad & \beta g_s(h_0,s^*,A_0) f_L(K'(\phi'),L'(\phi')) \\
& \times \int_{\eta'} u'\left(\begin{array}{l} k'^*(\phi,k_0,h_0;A_0)[1+f_K(K'(\phi'),L'(\phi'))] \\ +(1-s')\eta' g(h_0,s^*,A_0) f_L(K'(\phi'),L'(\phi')) \end{array}\right)\eta'\left\{1-s'+\frac{g_h(h',s',A_0)h'}{g_s(h',s',A_0)}\right\} f(\eta')d\eta' \\
+ \quad & (1-s^*)g_s(h_0,s^*,A_0)\beta\int_S\int_{\eta'} f(\eta')\left[\begin{array}{l} u'\Big(k'^*(\phi,k,h;A)[1+f_K(K'(\phi'),L'(\phi'))] \\ \quad +(1-s')\eta' g(h,s^*,A) f_L(K'(\phi),L'(\phi))-k''\Big) \\ \times\left\{\begin{array}{l} k'^*(\phi,k,h;A) f_{KL}(K'(\phi'),L(\phi')) \\ +(1-s')\eta' g(h,s^*,A) f_{LL}(K'(\phi'),L(\phi')) \end{array}\right\} \end{array}\right] d\eta'\phi dS = 0
\end{aligned}$$

That is, for all (k, h, A)

$$\begin{aligned}
& u'\Big(k[1+r(\Phi)]+(1-s^*(\Phi,k,h;A))hw(\Phi)-k'^*(\Phi,k,h;A)\Big) \\
= \quad & \beta g_s(h,s^*(\Phi,k,h;A),A)w(\Phi')E_{\eta'}\left[u'(c^*(\Phi,\Phi',k,h,\eta';A))\eta'\left\{1-s'+\frac{g_h(h',s',A)h'}{g_s(h',s',A)}\right\}\right] \\
+ \quad & (1-s^*)g_s(h,s^*(\Phi,k,h;A),A)\Delta_h, \quad \textit{where} \\
\Delta_h = \quad & \beta\int_S\int_{\eta'} f(\eta')\eta'\Big[u'\Big(k'[1+f_K(K'(\phi'),L'(\phi'))]+(1-s')h' f_L(K'(\phi),L'(\phi))-k''\Big) \\
& \times\Big\{k'^*(\phi,k,h;A) f_{KL}(K'(\phi'),L(\phi'))+(1-s')\eta' g(h,s^*,A) f_{LL}(K'(\phi'),L(\phi'))\Big\}\Big] d\eta'\phi dS
\end{aligned}$$

■