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Working Paper No. 60
October 1986.

RECURSIVE UTILITY AND THE RAMSEY PROBLEM

by

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October 1986

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I would like to thank Robert Becker and W. Davis Dechert for their helpful comments on earlier versions of this work.

Abstract

This paper examines the existence, continuity and characterization of optimal paths under Koopmans' "time-stationary" or "recursive" preferences. Such infinite-lived agents have flexible time preference. Their current utility is defined as a fixed function (the aggregator) of current consumption and future utility. Given a suitable aggregator, a useful refinement of the Contraction Mapping Theorem generates the utility function. A limiting argument, analogous to partial summation, constructs an upper semicontinuous utility function for an even broader class of aggregators. The classical Weierstrass method then demonstrates the existence of optimal paths. Under somewhat more stringent conditions on the aggregator and technology, optimal paths are continuous in the initial capital stocks, and are characterized through the Euler equations and a transversality condition.

1. Introduction

This paper examines the existence, continuity and characterization of optimal paths under time-stationary recursive preferences specified by an aggregator function. Given a suitable aggregator, a refinement of the Contraction Mapping Theorem generates the utility function. A limiting argument, analogous to partial summation, constructs an upper semicontinuous utility function for an even broader class of aggregators. The classical Weierstrass method then demonstrates the existence of optimal paths. Under somewhat more stringent conditions on the aggregator and technology, optimal paths are continuous in the initial capital stocks, and are characterized through the Euler equations and a transversality condition.

Recursive utility involves flexible time preference. In contrast, many dynamic economic models rely on the additively separable utility function. Unfortunately, a fixed rate of impatience can cause strange behavior in rather ordinary circumstances. A consumer facing a fixed interest rate will either try to save without limit, or borrow without limit, except in the knife-edge case where the rate of impatience equals the interest rate. This problem is especially severe when there are heterogeneous households. If all households face the same interest rate, and have different rates of impatience, they cannot all be in a steady state.

Recursive utility escapes this dilemma by allowing impatience to depend on income. These "recursive" or "time-stationary" preferences, introduced by Koopmans (1960), formalize a method used by Fisher (1930). Fisher employed a simple division between current consumption and future utility.[1] With recursive preferences, current utility is taken as a fixed function (the aggregator) of current consumption and future utility.

Koopmans' approach was to find axioms on preferences that would allow the utility function to be characterized by such an aggregator. I provide a converse in this paper. Given an aggregator, recursive substitution can be used to construct the utility function on an appropriate space of consumption paths. A new concept of impatience, β-myopia, helps build the link between the properties of the aggregator and the utility function. The parameter β combines information about both discounting and the asymptotic growth properties of the aggregator. This result applies to a wide variety of utility functions, including those with hyperbolic absolute risk aversion.[2]

Taking the aggregator as fundamental provides detailed information about preferences. First, it is a lot easier to specify an aggregator than a recursive utility function. Koopmans, Diamond and Williamson (1964) found an aggregator that had a specific property (increasing marginal impatience), but the corresponding utility function has never been explicitly computed. It does not have a closed form expression. Second, the aggregator, with its sharp distinction between current and future consumption, often makes it easier to incorporate hypotheses about intertemporal behavior. It can be quite difficult to translate axioms into usable conditions on the utility function. Further, if we impose behaviorial conditions as axioms, there is the question of their consistency. With aggregators, this is never a problem. Once the utility function exists, consistency is automatic. Finally, characterization of optima via the no-arbitrage conditions is simpler when we use the aggregator rather than the utility function or preference order.

As in Lucas and Stokey (1984), I construct the aggregator by means of a contraction mapping. Unfortunately, their method only applies to bounded utility functions. I introduce a refinement of the Contraction Mapping Theorem, the Weighted Contraction Theorem, which applies to a much broader class of utility functions that includes many standard examples. This theorem can also be used to lift the artificial restriction to bounded utility so common in stochastic dynamic models.[3] Utility functions that allow $-\infty$ as a value require further treatment. The contraction technique is combined with a "partial sum" technique to construct these utility functions.

Two additional results follow from the construction of the utility function. First, depending on its asymptotic marginal felicity, the aggregator can often be bounded by certain standard functions, which systematically determine the parameter β. Second, it enjoys useful continuity properties. As in Beals and Koopmans (1969), Majumdar (1975) and Magill and Nishimura (1984), the combination of continuity or upper semicontinuity with compactness of the feasible set immediately yields existence. Further, Majumdar showed that convexity properties can then be used to find support prices via the Hahn-Banach theorem. This is quite different from the routes taken by Gale (1967), Brock (1970) and Sutherland (1970). In particular, value loss techniques are, at best, difficult to apply to recursive preferences. Fortunately, continuity can still be used. As a bonus, Beals and Koopmans' route to existence is conceptually much simpler than the other approaches.

This method requires the choice of an appropriate commodity space. There are two conflicting tendencies here. When the space is small and the topology is strong, it is relatively easy to obtain a continuous utility function. The concept of β-myopia measures how far we must go. The space must also be large enough to contain the feasible set, and the topology weak enough to make it compact. In particular, the space must contain the path of pure accumulation.[4]

When the utility function is actually continuous, much more can be said.

The Maximum Theorem yields a continuous maximizer correspondence. In addition, with a concave utility function and convex technology, optima are characterized by a combination of the Euler equations and transversality condition. The weighted norms constructed to show existence are utilized in a squeezing argument that demonstrates the necessity of the transversality condition. They also help construct an approximate utility function used to show sufficiency of the transversality and Euler equations.

Of course, these techniques apply to the simplest recursive case, the additively separable utility function. As such, they bring many scattered results under one roof. They can deal with piecewise linear, time-varying (Brock and Gale, 1969), strongly productive (Gale and Sutherland, 1968), or nonclassical technologies (Dechert and Nishimura, 1980) as well as undiscounted (or even upcounted) utility in a unified way.

Section Two introduces the aggregator. The Weighted Contraction Theorem is proved in Section Three. The utility function is constructed in Section Four, using the Weighted Contraction Theorem and a "partial sum" approach. Optimal paths are studied in the next three sections. Existence is the subject of Section Five. Continuity is demonstrated in Section Six. Finally, the Euler equations and transversality condition characterize optimal paths in Section Seven. Concluding comments are gathered in Section Eight.

2. The Aggregator Approach

The formal analysis of recursive utility functions dates back to Koopmans (1960). In addition to demonstrating the axiomatic underpinnings of recursive utility, he showed how they could be constructed from an aggregator function W. The aggregator maps $\mathbf{X} \times \mathbf{Y}$ to $\mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are subsets of $\mathbb{R}_+ = \{x \in$

$\mathbb{R} : x \geq 0\}$.[5] Without loss of generality, I assume $0 \in \mathbf{Y}$. The aggregator W must take values in $\mathbf{Y}$ so that the recursion map T_W can be defined below. The recursive utility function will be the unique fixed point of T_W.

Possible aggregators include $W(x,y) = u(x) + \delta y$, which yields the additively separable utility function $\Sigma_{t=1}^{\infty} \delta^{t-1}u(x_t)$, Koopmans, Diamond and Williamson's (1964) aggregator $W(x,y) = (1/\theta) \log (1+\beta x^{\gamma}+\delta y)$, and the Uzawa aggregator $W(x,y) = (-1+y) \exp [-u(x)]$ as used by Uzawa (1968) and Epstein and Hynes (1983).

More formally, $W: \mathbf{X} \times \mathbf{Y} \rightarrow \mathbf{Y}$ is an *aggregator* if:

W1) W is continuous on $\mathbf{X} \times \mathbf{Y}$ and increasing in both x and y.

W2) W obeys a Lipschitz condition of order one, i.e., there exists $\delta > 0$ such that $|W(x,y) - W(x,y')| \leq \delta|y-y'|$ for all y, y' in $\mathbf{Y}$. (This is called uniformly bounded time perspective.)

Another useful property is

W3') W is jointly concave in x and y.

There are several important points about (W2). First, the Lipschitz bound δ is independent of x and y. Second, $\delta < 1$ is not required. This permits undiscounted or even upcounted models. Third, a variable δ could be handled, but would require much finer control over both the program space and the aggregator to give significant additional information. The uniform bound is chosen to make life simpler, and maintains a balance between power and ease of use.

Given such an aggregator, we can define the recursion map T_W. First, define the projection π and shift S by $\pi X = x_1$ and $SX = (x_2, x_3, \ldots)$ for $X \in \mathbb{R}^{\infty}$. The key property that makes a utility function U recursive is that $U(X) = W(\pi X, U(SX))$. Intuitively, we can find U by recursively substituting it in this equation. This recursive substitution is accomplished by the transformation T_W defined by $(T_W U)(X) = W(\pi X, U(SX))$. Thus $(T_W^N 0)(X) = W(x_1, W(x_1, \ldots, W(x_N, 0))\ldots))$.

Curiously, the aggregator need not be jointly concave for the associated utility function to be concave. The Uzawa aggregator is not concave, but the associated utility function $U(x) = -\Sigma_{t=1}^{\infty} \exp[-\Sigma_{\tau=1}^{t} u(x_\tau)]$ is concave. The Uzawa aggregator does satisfy condition (W3) below. Epstein (1983) gave sufficient conditions for the concavity of generalized Uzawa aggregators. Whenever the existence theorems of Section Four provide a utility function, condition (W3) is both necessary and sufficient for its concavity.

W3) $(T_W^N a)(X)$ is concave in X for all N and all constants $a \in \mathbf{Y}$.

LEMMA 1. Suppose $(T_W^N a)(X) \to U(X)$ *and (W3) or (W3′) holds. Then U is concave on its domain* $\mathbf{A}$. *If, in addition, W is strictly concave in x and strictly increasing in y, then U is strictly concave.*

PROOF. Suppose (W3′) holds, then (W3) holds. To see this, take $0 < \alpha < 1$ and $X, X' \in \mathbf{A}$, and assume $f(X) = [T_W^{N-1} a](X)$ is concave. Let $X^\alpha = \alpha X + (1-\alpha)X'$. Then $T_W f(X^\alpha) = W(x_1^\alpha, f(SX^\alpha)) \geq W(x_1^\alpha, \alpha f(SX) + (1-\alpha) f(SX')) \geq \alpha\, W(x_1, f(SX)) + (1-\alpha)\, W(x_1', f(SX'))$. By induction, (W3) follows. Finally, as the pointwise limit of concave functions, U is concave.

The argument for strict concavity is similar. QED

3. The Weighted Contraction Theorem

The Contraction Mapping Theorem is a useful technique for proving existence theorems and is often used in dynamic programming (e.g. Blackwell, 1965; Denardo, 1967). The commonly used forms of this theorem require bounded utility functions. Unfortunately, this rules out many of the usual utility functions. In particular, functional forms with constant elasticity of marginal utility, such as the logarithm, are not included. This is a severe restriction. Fortunately, a modification of the contraction technique can be used even when the utility functions are not bounded. This is the weighted contraction method.

Weighted function techniques are relatively new to dynamic programming (e.g., Wessels, 1977; Waldman, 1985),[6] and have only now found their way to economics. The idea is this: Instead of considering bounded functions, look at functions obeying a growth condition. This new function space has a natural norm induced by the growth condition. On this new (weighted) space, the mapping is a contraction.

Let $f \in C(\mathbf{A};\mathbf{B})$, the space of continuous functions from $\mathbf{A}$ to $\mathbf{B}$. Suppose $\varphi \in C(\mathbf{A};\mathbf{B})$ with $\mathbf{B} \subset \mathbb{R}$ and $\varphi > 0$. The function f is *φ-bounded* if $\|f\|_\varphi = \sup\{|f(x)|/\varphi(x)\} < \infty$, and the φ-norm is $\|f\|_\varphi$.[7] The φ-norm turns $C_\varphi(\mathbf{A};\mathbf{B}) = \{f \in C(\mathbf{A};\mathbf{B}) : f \text{ is } \varphi\text{-bounded}\}$ into a Banach space.[8] In particular, $C_\varphi(\mathbf{A};\mathbf{B})$ is a complete metric space. A transformation $T: C_\varphi \to C_\varphi$ is a *strict contraction* if $\|Tx-Ty\|_\varphi \leq \theta\|x-y\|_\varphi$ with $\theta < 1$. For such T, we have:

CONTRACTION MAPPING THEOREM. A strict contraction on a complete metric space has a unique fixed point.

The proof is well-known, and can be found in various standard references (e.g., Reed and Simon, 1972; Smart, 1974).

In applications, the main problem is to show that T is a strict contraction. An easy way to do this is by using monotonicity properties, as is common in dynamic programming. In the weighted contraction context, this yields the following form of the theorem.

WEIGHTED CONTRACTION MAPPING THEOREM (MONOTONE FORM). Let T: $C_\varphi \to C$ *such that*

1) *T is non-decreasing* ($u \leq v$ *implies* $Tu \leq Tv$).

2) $T(0) \in C_\varphi$.

3) $T(u + A\varphi) \leq Tu + A\theta\varphi$ *for some constant* $\theta < 1$ *and all* $A > 0$.

Then T has a unique fixed point.

PROOF. For all u, v $\in C_\varphi$, $|u-v| \leq \|u-v\|_\varphi\ \varphi$. So, $u \leq v + \|u-v\|_\varphi\ \varphi$ and $v \leq u + \|u-v\|_\varphi\ \varphi$. Properties (1) and (3) yield $Tu \leq Tv + \theta\|u-v\|_\varphi\ \varphi$ and $Tv \leq Tu + \theta\|u-v\|_\varphi\ \varphi$. Thus $\|Tu-Tv\|_\varphi \leq \theta\|u-v\|_\varphi$.

Setting v = 0, we have $\|Tu-T(0)\|_\varphi \leq \theta\|u\|_\varphi$, and so $\|Tu\|_\varphi \leq \theta\|u\|_\varphi + \|T(0)\|_\varphi < \infty$ by property (2). Hence T: $C_\varphi \to C_\varphi$. As $\theta < 1$, T is a strict contraction on C_φ. By the contraction mapping theorem, it has a unique fixed point. QED

This form of the Contraction Mapping Theorem is particularly adapted for use on dynamic programming problems, including economic models with a recursive structure. In particular, it can be used to establish the existence and continuity of recursive utility functions. It can also be used on stochastic programming problems such as those in Lucas' asset pricing model (1978). Other variants are possible. By using the techniques of Bhakta and Mitra (1984), the requirement that $\theta < 1$ can be relaxed. Further, a version of the Local Contraction Mapping Theorem can also be obtained.

4. The Existence of Recursive Utility

The recursive utility function will generally be defined on a space smaller than $\mathbb{R}^{\infty}$. Define $\mathbf{X}(\beta) = \{X \in \mathbb{R}_{+}^{\infty} : |X|_{\beta} < \infty\}$ where $|X|_{\beta} = \sup |x_t/\beta^{t-1}|$ is the β-weighted ℓ^{∞} norm. Here $\beta \geq 1$. The utility function will be a function on $\mathbf{X}(\beta)$ that is continuous in the topology generated by the β-norm (β-topology). These norms can be thought of as having the discount factor $1/\beta$ built in. Topologies of this type have been used by Chichilnisky and Kalman (1980) and Dechert and Nishimura (1980) to study optimal paths.

One notion of impatience is the concept of myopia introduced by Brown and Lewis (1981). Their basic idea was to use the continuity properties of the utility function to measure impatience. This notion was founded on the observation that Mackey continuity presupposes a certain degree of impatience. This idea may be further developed by considering continuity with respect to a variety of different topologies. In particular, we may use the β-topology. If U is defined and continuous on $\mathbf{X}(\beta)$, we call U *β-myopic*.

Let $\mathbf{A} \subset \mathbb{R}^{\infty}$ with $\pi(\cup_{N=0}^{\infty} S^N\mathbf{A}) \subset \mathbf{X}$. Both the shift S and projection π are continuous in any topology on $\mathbf{A}$ that is stronger than the relative product topology, as are the β-topologies. Given a positive function φ, continuous on $\mathbf{A}$, let $C = C(\mathbf{A};\mathbf{Y})$ and $C_{\varphi} = C_{\varphi}(\mathbf{A};\mathbf{Y})$. Since all the functions involved are continuous, $T_W: C_{\varphi} \to C$.

CONTINUOUS EXISTENCE THEOREM. Suppose the topology on $\mathbf{A}$ *is stronger than the relative product topology,* $W: \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ *obeys W1 and W2,* φ *is continuous,* $W(\pi X, 0)$ *is* φ*-bounded, and* $\delta\|\varphi \circ S\|_{\varphi} < 1$*. Then there exists a unique* $U \in C_{\varphi}$ *such that* $W(\pi X, U(SX)) = U(X)$*. Moreover,* $(T_W^N 0)(X) \to U(X)$*.*

PROOF. Since W is increasing in y, T_W is increasing. Now $|T_W(0)|/\varphi(X) = |W(x_1,0)|/\varphi(X) < \infty$ because $W(\pi X,0)$ is φ-bounded. Finally, $T_W(f + A\varphi) = W(x_1, f(SX) + A\varphi(SX)) \leq W(x_1,f(SX)) + A\delta\, \varphi(SX) \leq T_W f + A\delta\, \|\varphi \circ S\|_\varphi\, \varphi(X)$. The Weighted Contraction Theorem, with $\theta = \delta\|\varphi \circ S\|_\varphi < 1$, shows that T_W is a contraction, and has a unique fixed point U.

Finally, $\|U(X) - (T_W^N 0)(X)\|_\varphi \leq \delta^N \|U(S^N X)\|_\varphi \leq \|U\|_\varphi (\delta\|\varphi \circ S\|_\varphi)^N$. As the last term converges to zero, $(T_W^N 0)(X) \to U(X)$.

4.1. *Examples with Aggregator Bounded Below*

The easiest application of the Continuous Existence Theorem is to a bounded aggregator with $\delta < 1$ and $\mathbf{A} = \mathbf{X}(1)$. Take φ as the constant 1, and use the product topology. This yields a recursive utility function that is not only β-myopic for all $\beta \geq 1$, but also continuous in the relative product topology on $\mathbf{X}(1)$.[9] In particular, this applies to the Uzawa aggregator with $u(0) > 0$.

Another application is to W with $0 \leq W(x,0) \leq A(1 + x^\eta)$ as in the case where W(x,0) has asymptotic exponent or asymptotic elasticity of marginal felicity (see Brock and Gale, 1969) less than $\eta > 0$ with $\delta\beta^\eta < 1$. In this case, take $\mathbf{A} = \mathbf{X}(\beta)$ and $\varphi(X) = 1 + |X|_\beta^\eta$. Then $\|\varphi \circ S\|_\varphi = \beta^\eta$, and the recursive utility function is β-myopic. When $0 \leq W(x,0) \leq A(1 + \log x)$, a similar argument shows that U is β-myopic for all $\beta < \infty$.[10] In fact, when f is concave, $f(x) \leq f(1) + \alpha(x-1)$ for some supergradient α. (If differentiable, $\alpha = f'(1)$.) Taking $\varphi(X) = 1 + |X|_\beta$ shows that any aggregator that is concave in x yields a β-myopic utility function whenever $\delta\beta < 1$.

Whether $W(0,0) = 0$ is really a matter of convenience. The important fact used is that W is bounded below. If $W(0,0) \neq 0$, a modest adjustment of the utility scale can remedy the situation. Use the adjusted aggregator, $V(z,y) =$

$W(z, y + U(0)) - U(0)$. Both aggregators generate equivalent utility functions, and $V(0,0) = 0$. When applied to the Uzawa aggregator $W(z,y) = (-1 + y)\, e^{-u(z)}$, this yields $U(0) = W(0,U(0)) = (-1 + U(0))\, e^{-u(0)}$, so $U(0) = 1/(1-v)$ where $v = e^{u(0)}$. The adjusted aggregator is then $V(z,y) = [y + v/(1-v)]\, e^{-u(z)} - 1/(1-v)$.

In general, the condition $\beta^{\eta}\delta < 1$ cannot be relaxed without losing existence on $\mathbf{X}(\beta)$. Let $W(z,y) = z^{\eta} + \delta y$ and take $\beta = \delta^{-1/\eta}$. The utility function cannot be defined when X is given by $x_t = \beta^t$. No utility function can be constructed from the aggregator on $\mathbf{X}(\beta)$. A smaller space must be used.

4.2. *Unbounded Aggregators*

The Continuous Existence Theorem can also be used indirectly to deal with aggregators that are not bounded below. These aggregators have $W(0,0) = -\infty$. Paths that are near 0 can also pose problems. For this reason, we use a donut-shaped region of $\mathbb{R}^{\infty}$ for the set $\mathbf{A}$. Choose $\gamma \leq \beta$, and define ${}_{\gamma}|X| = \inf |x_t/\gamma^{t-1}|$ if $0 < \gamma < \infty$ and ${}_{\gamma}|X| = 0$ if $\gamma = 0$. Then take $\mathbf{A} = \mathbf{X}(\beta,\gamma) = \{X \in \mathbb{R}_{+}^{\infty} : |X|_{\beta}, {}_{\gamma}|X| < \infty\}$. This is the set of paths that have a growth rate between γ and β. For $\beta = \infty$, set $|X|_{\beta} = 0$. Thus $\mathbf{X}(\beta,0)$ is just our old friend $\mathbf{X}(\beta)$ while $\mathbf{X}(\infty,\gamma)$ is the space of paths with growth rates at least γ. Note that $\mathbf{X}(\infty,0) = \mathbf{X}(\infty) = \mathbb{R}_{+}^{\infty}$. Suppose there are increasing functions g and h with $g(x) \leq W(x,0) \leq h(x)$ and $\varphi > 0$ where $\varphi(X) = \max\{h(|X|_{\beta}), -g({}_{\gamma}|X|)\}$. Then $W(x,0)$ is φ-bounded and trivially continuous in the discrete topology. If $\delta\|\varphi \circ S\|_{\varphi} < 1$, the Existence Theorem will apply. This yields a utility function defined on $\mathbf{X}(\beta,\gamma)$. Unless $\gamma = 0$, this does not directly yield any useful type of continuity. However, it is the first step of an indirect attack on the problem.

SEMICONTINUOUS EXISTENCE THEOREM. *Suppose* $W: \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ *obeys W1 and W2, g, h and* φ *are as above,* $W(\pi X, 0)$ *is* φ*-bounded, and* $\delta\|\varphi \circ S\|_\varphi < 1$. *Then there exists an U obeying* $W(\pi X, U(SX)) = U(X)$ *that is upper semicontinuous in the* β*-topology on* $\mathbf{X}(\beta)$ *if* $\beta < \infty$ *and upper semicontinuous in the product topology on* $\mathbf{X}(\infty)$ *if* $\beta = \infty$.

PROOF. For $\gamma > 0$ and $\beta < \infty$, let $b_t = \beta^{t-1}$ and define the "partial sums" on $\{X \in \mathbf{X}(\beta,\gamma) : |X|_\beta \le k\}$ by

$$S_N(x) = T_W^N U(kS^N B)(X) = W(x_1, W(x_2, \ldots, W(x_N, U(kS^N B))\ldots)).$$

Since $x_{N+1} \le k\beta^N$ and W is increasing, $W(x_{N+1}, U(kS^{N+1}B)) \le W(k\beta^N, U(kS^{N+1}B)) = U(kS^N B)$. Substituting this in $S_N(X)$ yields $S_{N+1}(X) \le S_N(X) \le U(kB)$. Thus U(X) given by $\lim_{N\to\infty} S_N(X) = \inf_N S_N(X)$ is either finite or $-\infty$.

Since U is the infimum of functions that are upper semicontinuous in the β-topology, it is upper semicontinuous in the β-topology when $|X|_\beta \le k$.[11] Provided U is well-defined on $\mathbf{X}(\beta)$, it is upper semicontinuous there since any β-convergent sequence is also β-bounded. When $\beta = \infty$ and $W(\infty,0) = 0$, interpreting $U(S^N B)$ as 0 gives the same result on all of $\mathbf{X}(\infty)$, while any W with $W(\infty,0) < \infty$ can be transformed so that $W(\infty,0) = 0$.

The bounds obtained in the Continuous Existence Theorem show that U is well-defined and recursive. The same estimate can be used to prove both of these facts. Write $S_{N,k}$ to denote the dependence on k. For $k' > k$,

$$|S_{N,k}(X) - S_{N,k'}(X)| \le \delta^N |U(kS^N B) - U(k'S^N B)|$$

$$\le \delta^N M [\varphi(kS^N B) + \varphi(k'S^N B)]$$

$$\leq \; M' \; (\delta \|\varphi \circ S\|_{\varphi})^{N}$$

For some M'. The first step uses the Lipschitz bound (W2). The second uses the φ-boundedness of U on $\mathbf{X}(\beta,\gamma)$, and the third uses fact that $\varphi(S^N X) \leq (\|\varphi \circ S\|_{\varphi})^N \varphi(X)$. As this last expression converges to zero, the definition of U is independent of k. Now set $k' = k\beta$. Recursivity now follows from $W(x_1, S_{N,k'}(SX)) = W(x_1, \ldots, W(x_{N+1}, U(kS^{N+1}B))\ldots) = S_{N+1,k}(X)$ since letting $N \to \infty$ and using the continuity of W shows $W(x_1, U(SX)) = U(X)$. QED

One point of importance is that the utility function exists on $\mathbf{X}(\beta,\gamma)$. Further, it is φ-bounded there, and thus finite. Unfortunately, the Continuous Existence Theorem does not yield useful information about the continuity of U on $\mathbf{X}(\beta,\gamma)$ as we were forced to use the discrete topology, where all functions are continuous. Fortunately, upper semicontinuity suffices for demonstrating the existence of optimal paths.

Aggregators with $-1 - \min\{0, \log x\} \leq W(x,0) \leq C + \log(1+x)$ fall into this framework. Given $\delta < 1$ and $\beta \geq 1$, the constant C may be assumed large enough that $\delta(C + \log\beta)/C < 1$. Taking $\gamma = 1$ shows that utility function exists on $\mathbf{X}(\beta,1)$ for any β. In other cases, upcounting ($\delta > 1$) may be allowed. When $-x^{\eta} \leq W(x,0) \leq 0$ with $\eta < 0$, we require $\delta\beta^{\eta} \leq \delta\gamma^{\eta} < 1$. As $\eta < 0$, $\beta^{\eta} < 1$ and there are γ that permit $\delta > 1$. The Upper Semicontinuous Existence Theorem applies to these examples.

The "partial sum" approach works on a wider range of aggregators than considered in the theorem. One of these is the Rawlsian aggregator, $W(z,y) = \min\{z,y\}$. Note that $\delta = 1$. In this case, the "partial sums" $S_N(x) = \inf\{x_t : t \leq N\}$ are independent of k and β. Recursivity follows instantly. Further, the "partial sums" are defined on all of $\mathbb{R}^{\infty}_{+}$ and continuous in the product topology there. This means that the Rawlsian utility function U(x) =

$\inf x_t$ is upper semicontinuous in the product topology. The same is true of the discounted or upcounted versions $W(z,y) = \min\{z, \delta y\}$.

If there is a function $\upsilon(x)$ with $\upsilon(x) = W(x,\upsilon(x))$, "partial sums" can be defined by $[T_W^N \upsilon(k)](X)$. These form a decreasing sequence, so their limit is an upper semicontinuous function $U(X)$. As $W(x_1, U(SX)) = \lim [T_W^{N+1}\upsilon(k)](X) = U(X)$, this yields a recursive utility function. This recursive utility function may fail to be lower semicontinuous. One such example is $W(x,U) = -1 + e^{-x} U$. Here $\upsilon(x) = -1/(1 - e^{-x})$, and $U(X) = \Sigma_{t=1}^{\infty} \exp(-\Sigma_{\tau=1}^{t} x_\tau)$. Setting $x_t = 2 \log (t+1)/t$, and taking the sequence $X^n = (x_1, \ldots, x_n, 0, \ldots)$, shows that this utility function is not lower semicontinuous since $U(X^n) = -\infty$ but $U(X) > -\infty$. Note that $\delta = 1$ in this example.[12]

5. The Existence of Optimal Paths

The existence of optimal paths is shown by a version of the classical Weierstrass method. Upper semicontinuous functions defined on compact sets have maxima.

OPTIMAL PATH THEOREM. If **F** *is a* β*-compact subset of* $\mathbf{X}(\beta)$*, and U is* β*-upper semicontinuous on* $\mathbf{X}(\beta)$*, there exists a* $C^* \in \mathbf{F}$ *with* $U(C^*) = \sup\{U(C) : C \in \mathbf{F}\}$.

PROOF. Let $U^* = \sup\{U(C) : C \in \mathbf{F}\}$. Take a maximizing sequence C_n so $U(C_n) \to U^*$. Since **F** is β-compact, there is a subsequence, also denoted C_n that β-converges to some feasible C^*. By the β-upper semicontinuity of U, $U(C^*) \geq \lim U(C_n) = U^*$. But $U(C^*) \leq U^*$, so $U(C^*) = U^*$. QED

LEMMA 2. Let $\alpha < \beta$ and let **F** *be an α-bounded set. The β-topology and the relative product topology coincide on* **F**.

Proof. Let k be an α-bound for **F**. Clearly the β-topology is stronger than the product topology on **F**. Suppose $X^n \to X$ in the product topology. Given $\epsilon > 0$, choose M such that $k(\alpha/\beta)^M < \epsilon$. Since $X^n \to X$, we can find an N with $\sup_{t \leq M} \{|x_t^n - x_t|/\beta^t\} < \epsilon$ for $n > N$. But then $|X^n - X|_\beta < 2\epsilon$ for $n > N$, and $X^n \to X$ in the β-topology. The two topologies are identical. QED

The lemma is useful when trying to verify that a set is actually β-compact. If it is α-bounded for some $\alpha < \beta$, it suffices to show the set is compact in the relative product topology. As a general rule, the product topology is easier to work with. It is crucial that $\beta > \alpha$. Majumdar (1975) gives an example illustrating why norm-bounded feasible sets are not compact in the norm topology. The same sort of problem would occur here if $\beta = \alpha$. In fact, $\beta = \alpha = 1$ is precisely Majumdar's case.

One application is to a one-sector model of optimal capital accumulation (Ramsey model). In the classical Ramsey model, the technology is described by a (gross) *production function*. The production function f is a continuous, non-decreasing function $f: \mathbb{R}_+ \to \mathbb{R}_+$. Note that $f(0) \geq 0$. In the time-varying Ramsey model, the technology is described by a sequence, $\{f_t\}$, of such production functions. Given this production technology, the set of feasible paths of accumulation from initial stock k (the *production correspondence*) is $\mathbf{P}(k) = \{X \in \mathbf{X}(\infty) : 0 \leq x_t \leq f_t(x_{t-1}),\ x_0 = k\}$. The set of feasible paths, F(k) is $\{C \in \mathbf{X}(\infty) : 0 \leq c_t \leq f_t(x_{t-1}) - x_t$ for some $X \in \mathbf{P}(k)\}$. Both **F**(k) and **P**(k) are closed in the product topology and $\mathbf{F}(k) \subset \mathbf{P}(k) \subset X_{i=1}^{\infty} [0, f^t(k)]$. As this last set is compact by Tychonoff's Theorem, **F**(k) is also compact in the product topology. Define f^t inductively by $f^1 = f_1$ and $f^t = f_t \circ f^{t-1}$. The *path of pure accumulation* is $\{f^t(k)\}$. If $\lim_{t \to \infty} [f^t(k)/\alpha^t] < \infty$, both $\mathbf{P}(k)$ and $\mathbf{F}(k)$ are α-bounded subsets of $\mathbf{X}(\beta)$. When $\mathbf{P}(k)$ is α-bounded, we call the technology α-bounded. By Lemma 2, any α-bounded technology yields a β-compact feasible set for $\beta > \alpha$. The Optimal Path Theorem then shows that an optimal path exists whenever U is β-upper semicontinuous on $\mathbf{X}(\beta)$ for some $\beta > \alpha$.

Many commonly used production functions yield α-bounded technologies. The Cobb-Douglas production function with depreciation parameter λ is $f(x) = Ax^{\rho} + (1-\lambda)x$ where $A > 0$, $0 < \rho < 1$, and $0 \leq \lambda \leq 1$. For this, take $\alpha = 1$ when $\lambda > 0$ and any $\alpha > 1$ when $\lambda = 0$. For the affine technology $f(x) = Ax + B$ with $A > 0$ and $B \geq 0$, take $\alpha = A$ when $B = 0$ and any $\alpha > A$ when $B > 0$. Production functions of the form $f(x) = (1 + \sqrt{x})^2$ can use any $\alpha > 1$. As any concave production function obeys $f(x) \leq f(a) + \xi(x-a)$ for some supergradient ξ (e.g. $\xi = f'(a)$), it is α-bounded for any $\alpha > \xi$. Thus, any stationary, concave, production technology is α-bounded for all $\alpha > \lim_{k \to \infty} f'(k)$. A time-varying example is the case of exogenous technical progress where $f_t(x) = e^{nt}x^{\rho}$. The path of pure accumulation grows at asymptotic rate $\exp\{n/(1-\rho)\}$, so the technology is α-bounded for $\alpha > \exp\{n/(1-\rho)\}$.

Yet another type of feasible set is a budget set. This may admit consumption paths that grow too fast. However, if we define utility to be infinite on such paths, the choice of such paths may not be feasible in equilibrium. For the budget sets considered in Becker, Boyd and Foias (1986), it is enough that the utility function be well-behaved on technically feasible paths. They approximate the utility function by functions that are continuous on $\mathbb{R}^{\infty}$, and demonstrate existence of equilibrium for this modified economy. A limiting argument then shows that paths with infinite utility are not in the budget set in equilibrium.[13]

6. Sensitivity of Optimal Paths

The existence of optimal paths is just one of the useful facts that follow from continuity of the utility function and compactness of the feasible set. When the aggregator defines a continuous utility function, a modern version of Weierstrass' theorem, the Maximum Theorem, can be used to show continuity of optimal paths. For example, when the budget set (and hence the optimal path) depends continuously on a parameter vector ω, the maximizer correspondence $\mathbf{M}(\omega)$ will be continuous.

Maximum Theorem. If U is β-continuous and $\mathbf{F}(\omega)$ *is β-lower semicontinuous in ω and β-compact-valued, then the value function* $J(\omega) = \sup u(\mathbf{F}(\omega))$ *is continuous and the maximizer* $\mathbf{M}(\omega)$ *is upper semicontinuous. Further, if U is strictly concave, then* $\mathbf{M}(\omega)$ *is a continuous function of ω.*

Proof. This is a form of the Maximum Theorems found in Berge (1963) or Klein and Thompson (1984).

In the Ramsey model with an α-bounded technology, $\mathbf{F}(k)$ is β-compact and continuous for $\beta > \alpha$. Since $\mathbf{F}(k)$ is α-bounded, the β and product topologies are equivalent on $\mathbf{F}(k)$. In the product topology, the set of feasible paths is the continuous image of the production correspondence, so it is enough to show that the production correspondence $\mathbf{P}(k)$ is product continuous.

Upper semicontinuity is easy. The production correspondence is closed since each of the inequalities that define it depend on only finitely many periods, and are preserved under limits. Further, for k' near k, $\mathbf{P}(k') \subset$

$\mathbf{P}(k+1)$. Locally, everything takes place in a compact set. Thus closedness implies upper semicontinuity.

For lower semicontinuity, it is enough to show lower semicontinuity for the subbasic open sets $G(Y,\epsilon,N) = \{K \in \mathbb{R}_+^\infty : |x_t - y_t| < \epsilon \text{ for all } t < N\}$. Let $\epsilon, N > 0$ be given. Take $Y \in \mathbf{P}(k)$. By continuity of the f_t, we can choose δ with $|f^t(k') - f^t(k)| < \epsilon$ for all $t \leq N$ when $|k-k'| < \delta$. For any such k', take the path $x_t = \min \{y_t, f^t(k')\}$. Note that $f^t(k') + \epsilon > f^t(k) \geq y_t$ for $t \leq N$, so $y_t \geq x_t > y_t - \epsilon$ for all $t \leq N$. Hence $X \in G(Y,\epsilon,N)$. Further, $f_{t+1}(x_t) = \min \{f_{t+1}(y_t), f^{t+1}(k')\} \geq x_{t+1}$ and $x_1 \leq f_1(k')$, so $X \in \mathbf{P}(k')$. It follows $\mathbf{P}(k') \cap G(Y,\epsilon,N) \neq \emptyset$ whenever $|k-k'| < \delta$, establishing lower semicontinuity. Therefore $\mathbf{P}$ is a continuous correspondence.

An immediate application is to demonstrate β-continuity of optimal paths as a function of initial capital stock. One consequence is that $c_t(k)$ is continuous in k for each t. In general, this only holds for $\beta > \alpha$. For $\beta = \alpha$, it can fail even in models with additively separable utility. Amir, Mirman and Perkins (1983) and Dechert and Nishimura (1983), using a non-convex stationary technology, find that optimal paths converge to zero if the initial capital stock is below some critical value. The critical value is itself a steady state, and the only optimal path from the critical value is to remain there. Above the critical value, optimal paths converge to a steady state that lies above the critical value. They assume a maximum sustainable stock, so $\alpha = 1$ will do. The optimal path is clearly not norm ($\alpha = 1$) continuous.

Variations on this are possible. Stronger forms of the maximum theorem allow the utility function to depend on the parameter ω. If the bounds of Section Four hold uniformly in ω, the optimal paths will be continuous in ω.[14] A simple example is an optimal growth model with additively separable utility $W(z,y) = u(z) + \delta y$. Take $(k,\delta) = \omega \in \Omega = \mathbb{R}_+ \times [0,\delta]$ with a strictly concave, bounded u and $\delta < 1$. With a stationary concave production function f, a unique optimal path $\{c_t(k,\delta)\}$ exists. Further, $\{c_t(k,\delta)\}$ is β-continuous, hence $c_t(k,\delta)$ is a continuous function of (k,δ) for all $(k,\delta) \in \Omega$.

When the turnpike property holds, β-continuity of optimal paths will imply α-continuity. In fact, if optimal paths starting in some interval of initial stocks converge to the same steady state, α-continuity follows on that interval. Beals and Koopmans (1969) and Magill and Nishimura (1984) have demonstrated how these properties follows from monotonicity of the optimal paths. Beals and Koopmans examined conditions where a convex technology would yield monotonic optimal paths. A necessary and sufficient condition for monotonicity is not known in general. However, in the additively separable case, Dechert and Nishimura (1983) carried out an analysis of monotonicity in a reduced form model. In the general aggregator case, the analogous reduced form model has yet to be constructed.

7. Characterization of Optimal Paths

Optimal paths for the Ramsey model are characterized in this section. A useful envelope theorem and the Euler equations are developed first. I then proceed to the main result that the Euler equations, together with the transversality condition, completely characterize optimal paths for a large class of aggregators.

The following assumptions will be maintained throughout this section. The recursive utility function U obeys $U(0) = 0$ and is concave and φ-bounded on $\mathbf{X}(\beta)$ for some φ with $\|\varphi \circ S\|_\varphi < 1/\delta$. In addition, the feasible set $\mathbf{F}$ is generated by an α-bounded technology for some $\alpha < \beta$ given by a sequence continuous, concave, increasing production functions $\{f_t\}$ with $f_t(0) = 0$. As a consequence, the theorems of Sections Five and Six apply. The value function $J(y)$ is defined and continuous in the initial income $y = f_1(k)$.

Envelope Theorem. The value function J is increasing and concave. If U is differentiable with respect to consumption in period 1, then J is differentiable and obeys $J'(y) = U_1(C)$ where C is any optimal path from y.

Proof. The first two properties follow from the usual arguments. Differentiability is established as follows.[15] Let $h > 0$, $H = (h,0,\ldots0)$, and let C be an optimal path with initial income y so that $J(y) = U(C)$. Clearly, $J(y+h) \geq U(C+H)$ and thus $J(y+h) - J(y) \geq U(C+h) - U(C)$. Dividing by h and taking the limit shows that the right-hand derivative $J'(y+)$ satisfies $J'(y+) \geq U_1(C)$. Repeating this with $h < 0$ shows $J'(y-) \leq U_1(C) \leq J'(y+)$. As J is concave, $J'(y+) \leq J'(y-)$, thus $J'(y) = U_1(C)$. QED

Corollary. If U is recursive, and the aggregator is differentiable, then $J'(y) = W_1(c_1,U(SC))$ where C is any optimal path from y.

Before proceeding, note that partial derivative of U with respect to consumption at time t is given by

$$U_t(C) = W_2(c_1,U(SC))\, W_2(c_2,U(S^2C)) \ldots W_2(c_{t-1},U(S^{t-1}C))\, W_1(c_t,U(S^tC)).$$

Define the *marginal rate of impatience* R by

$$1 + R(C) = W_1(c_1,U(SC))/W_2(c_1,U(SC))\, W_1(c_2,U(S^2C)).$$

Then we have $U_t(C)/U_{t+1}(C) = 1 + R(S^{t-1}C)$.

Let C^* be optimal and let X^* be the associated sequence of capital stocks. Define $C_s: \mathbb{R}_+ \to \mathbb{R}^\infty$ by $c_{s,t}(x) = c^*_t$ for $t \neq s$, $s+1$; $c_{s,s}(x) = f_t(x^*_{t-1}) - x$; $c_{s,s+1}(x) = f_{t+1}(x) - x^*_{t+1}$. Then x^*_t solves $\max\ \{U(C_t(x)) : 0 \leq x \leq f_t(x^*_{t-1})\}$.

By the Kuhn–Tucker theorem,

$$U_{t+1}(C^*)\ f'_{t+1}(x_t^*) - U_t(C^*) \begin{cases} \geq 0 & \text{if} \quad x_t^* = f_t(x_{t-1}^*) \\ = 0 & \text{if} \quad 0 < x_t^* < f_t(x_{t-1}^*) \\ \leq 0 & \text{if} \quad x_t^* = 0. \end{cases}$$

Equivalently,

$$f'_{t+1}(x_t^*) \begin{cases} \geq & 1 + R(S^{t-1}C^*) \quad \text{if} \quad x_t^* = f_t(x_{t-1}^*) \\ = & 1 + R(S^{t-1}C^*) \quad \text{if} \quad 0 < x_t^* < f_t(x_{t-1}^*) \\ \leq & 1 + R(S^{t-1}C^*) \quad \text{if} \quad x_t^* = 0. \end{cases}$$

In either form, these will be referred to as the *Euler equations*.[16] The Euler equations will be instrumental in proving the Transversality Theorem.

Transversality Theorem. A path C^* *is optimal if and only if the Euler equations hold and* $\lim_{t \to \infty} x_t U_t(C^*) = 0$.

Proof. Suppose C^* is optimal.[17] As above, the optimal path must satisfy the Euler equations. Let $y_t = f_t(x_{t-1})$ denote the income stream associated with the optimal path C^* and J_t denote the value function at time t. As $J_t(0) = 0$, and J_t is concave, $J_t(y) \geq y\, J_t'(y)$ for all $y \geq 0$. Setting $y = y_t$ yields

$$J_t(y_t) \;\geq\; y_t\ W_1(c_t, U(S^tC)) \;\geq\; x_t\ W_1(c_t, U(S^tC)). \tag{1}$$

Now $J_t(y_t) = U(S^{t-1}C)$. Multiplying through by δ^{t-1} and using eq. (1) yields

$$\delta^{t-1}\ U(S^{t-1}C) \;\geq\; x_t\ U_t(C) \;\geq\; 0$$

Combining the φ-boundedness of U with $\delta\|\varphi \circ S\|_{\varphi} < 1$ shows $\lim x_t U_t(C) = 0$.

For the other half, let C be feasible and define the approximate utility function by $G_n(C) = [T_W^n U(C^*)](C)$. Thus $G_n(C) \to U(C)$. Further, G_n is concave and $G_n(C^*) = U(C^*)$, so

$$U(C^*) - G_n(C) \geq \Sigma_{t=1}^{n} U_t(C^*)(c_t^* - c_t).$$

Now $c_t^* - c_t = f_t(x_{t-1}^*) - f_t(x_{t-1}) - x_t^* + x_t$. Since f is concave, $c_t^* - c_t \geq f_t'(x_{t-1}^*)[x_{t-1}^* - x_{t-1}] - x_t^* + x_t$. Substituting and rearranging yields

$$\begin{aligned}
U(C^*) - G_n(C) &\geq \Sigma_{t=1}^{n-1} U_{t+1}(C^*)\, f_{t+1}'(x_t^*)\, [x_t^* - x_t] - \Sigma_{t=1}^{n} U_t(C^*)[x_t^* - x_t] \\
&\geq \Sigma_{t=1}^{n-1} [U_{t+1}(C^*)\, f_{t+1}'(x_t^*) - U_t(C^*)][x_t^* - x_t] + U_n(C^*)[x_n - x_n^*] \\
&\geq - U_n(C^*)x_n^*.
\end{aligned}$$

The last inequality follows from the Euler equations, which imply that all of the terms in the summation are non-negative. Letting $n \to \infty$ shows that $U(C^*) \geq U(C)$ for all feasible C. Thus C^* is optimal. QED

8. Conclusion

A number of extensions are possible. Similar techniques will work when there are many commodities available at each time. In fact, a Banach space of commodities at each time can be dealt with in similar fashion.[18] A time-varying aggregator (Streufert, 1985) can also be used, resulting in a recursive type of variable discount rate model (McKenzie, 1974; Mitra, 1979).

Other possible developments involve the weighted contraction and "partial sum" techniques. Stochastic models with recursive preferences have gotten attention as of late (Epstein, 1983; Bergman, 1985; Nairay, 1985). The Weighted Contraction Theorem may prove fruitful for investigating stochastic or continuous-time models such as the Lucas (1978) asset pricing model. The "partial sum" approach suggests how the concepts of "overtaking" and "catching up" can be extended to recursive preferences. In fact, when a program with $u = -\infty$ is compared with programs with finite u, an implicit overtaking criterion is present. Exactly how overtaking can be applied in a general recursive framework remains to be seen.

Further work also needs to be done on relating the new β-myopia concept to more traditional notions of impatience based on the marginal rate of substitution for consumption in adjacent time periods.

Footnotes

1. Fisher actually referred to "real income", but he also emphasized that income would ideally be measured in terms of utility.

2. Previous work along these lines (Lucas and Stokey, 1984) only applied to bounded aggregators.

3. Examples where it may prove useful include Lucas (1978), Prescott and Mehra (1980) and Danthine and Donaldson (1981). A weakness of these (and other) papers is that explicitly solvable examples frequently involve unbounded utility, while the general theory only applies to bounded utility. Weighted contractions can remedy this.

4. This choice of the commodity space is similar in motivation to the choice of both commodity and price spaces in Aliprantis, Brown and Burkinshaw (1985). In their terminology, this commodity space is the Riesz ideal generated by the path of pure accumulation.

5. The following notational conventions will be used. Sets are denoted by boldface capitals, vectors in $\mathbb{R}^{\infty}$ by capitals and real numbers by lowercase.

6. Weighted function spaces have a long history in the Fourier Analysis literature, e.g., Stein (1956).

7. This type of weighting must be distinguished from another type of weighting previously used in economics. The second type of weighting deals with the underlying commodity space rather than functions on it. Examples include Chichilnisky (1977, 1981) and Magill (1981) for continuous-time models and Chichilnisky and Kalman (1980) and Dechert and Nishimura (1980) for discrete-time models.

8. Let Vf = fφ. Since φ is continuous, Vf is a continuous function whenever f is continuous. Further, V is an isometric isomorphism from the Banach space of bounded continuous functions to $C_\varphi(\mathbf{A};\mathbf{B})$. Hence $C_\varphi(\mathbf{A};\mathbf{B})$ is also a Banach space.

9. This product continuity was the result obtained by Lucas and Stokey (1984).

10. Take $C > 0$ such take $\delta(C + \log \beta)/C < 1$ and set $\varphi(x) = C + \log(1+x)$.

11. A function f is *upper semicontinuous* if $\{f(x) < a\}$ is open for all a.

12. Some aggregators that seem to have $\delta = 1$ are really well-behaved. One example, used in Streufert (1986), is $W(x,U) = x + \sqrt{U}$, where $W_2 = 1/2\sqrt{U}$. This blows up as $U \to 0$. However, if $U \geq 1$, $W \geq 1$, so we take $\mathbf{Y} = [1,\infty)$. On this set, $\delta = 1/2$, and all is well.

13. In a different setting, Peleg and Yaari (1970) also utilized the fact that paths with infinite utility were outside the budget set in equilibrium.

14. Details may be found in Boyd (1986).

15. This method is adapted from Mirman and Zilcha (1975).

16. When the Inada condition $\lim_{c \to 0} W_1(c, U) = \infty$ holds, consumption will be strictly positive on the optimal path. Since $f_t(0) = 0$, the associated path of capital stocks is also positive. The inequalities reduce to equality under the Inada condition. This is not needed in the text.

17. This portion of the proof is based on the squeezing argument put forth in Mirman and Zilcha (1975) and corrected by Becker (1985).

18. Just define $|X|_\beta = \sup \|x_t/\beta^t\|$ and $_\gamma|X| = \inf \|x_t/\gamma^t\|$ where $\|\cdot\|$ is the Banach norm.

References

Aliprantis, C. D., Donald J. Brown and O. Burkinshaw (1985), Edgeworth Equilibria, Cowles Foundation Discussion Paper #756, Yale University.

Amir, Rabah, Leonard J. Mirman and William R. Perkins (1983), Nonclassical Optimal Growth, University of Illinois Discussion Paper.

Beals, Richard and Tjalling C. Koopmans (1969), Maximizing Stationary Utility in a Constant Technology, *SIAM J. Appl. Math.* 17, 1001-1015.

Becker, Robert A. (1985), Capital Income Taxation and Perfect Foresight, *J. Pub. Econ.* 26, 147-167.

Becker, Robert A., John H. Boyd III and Ciprian Foias (1986), The Existence of Ramsey Equilibrium, Working Paper, Indiana University.

Berge, C. (1963), *Topological Spaces*, Macmillan, New York.

Bergman, Yaacov Z. (1985), Time Preference and Capital Asset Pricing Models, *J. Financial Econ.* 14, 145-159.

Bhakta, P. C. and Sumitra Mitra (1984), Some Existence Theorems for Functional Equations Arising in Dynamic Programming Problems, *J. Math. Anal. Appl.* 98, 348-362.

Blackwell, David (1965), Discounted Dynamic Programming, *Ann. Math. Stat.* 36, 225-235.

Boyd, John H., III (1986), *Preferences, Technology and Dynamic Equilibria*, PhD. Dissertation, Indiana University.

Brock, William A. (1970), On Existence of Weakly Maximal Programmes in a Multi-Sector Economy, *Rev. Econ. Stud.* 37, 275-280.

Brock, William A. and David Gale (1969), Optimal Growth under Factor Augmenting Progress, *J. Econ. Theory* 1, 229-243.

Brown, Donald J. and Lucinda M. Lewis (1981), Myopic Economic Agents, *Econometrica* 49, 359-368.

Chichilnisky, Graciela (1977), Nonlinear Functional Analysis and Optimal Economic Growth, *J. Math. Anal. Appl.* 61, 504-520.

Chichilnisky, Graciela (1981), Existence and Characterization of Optimal Growth Paths Including Models with Non-Convesities in Utilities and Technologies, *Rev. Econ. Stud.* 48, 51-61.

Chichilnisky, Graciela and P. J. Kalman (1980), Application of Functional Analysis to Models of Efficient Allocation of Economic Resources, *J. Opt. Theory Appl.* 30, 19-32.

Danthine, J.-P. and John B. Donaldson (1981), Stochastic Properties of Fast vs. Slow Growing Economies, *Econometrica* 49, 1001-1033.

Dechert, W. Davis and Kazuo Nishimura (1980), Existence of Optimal Paths and the Turnpike Property: The Non-Convex Case with Unbounded Stocks, Working Paper, SUNY-Buffalo.

Dechert, W. Davis and Kazuo Nishimura (1983), A Complete Characterization of Optimal Growth Paths in an Aggregated Model with a Non-Concave Production Function, *J. Econ. Theory* 31, 332-354.

Denardo, Eric V. (1967), Contraction Mappings in the Theory Underlying Dynamic Programming, *SIAM Rev.* 9, 165-177.

Epstein, Larry G. (1983), Stationary Cardinal Utility and Optimal Growth Under Uncertainty, *J. Econ. Theory* 31, 133-152.

Epstein, Larry G. and J. Allan Hynes (1983), The Rate of Time Preference and Dynamic Economic Analysis, *J. Pol. Econ.* 91, 611-635.

Fisher, Irving (1930), *The Theory of Interest*, Macmillan, New York.

Gale, David (1967), On Optimal Development in a Multi-Sector Economy, *Rev. Econ. Stud.* 34, 1-18.

Gale, David and W. R. S. Sutherland (1968), Analysis of a One Good Model of Economic Development, in *Mathematics of the Decision Sciences: Part 2* (G. B. Dantzig and A. F. Veinott, eds.), American Mathematical Society, Providence.

Klein, Erwin and Anthony C. Thompson (1984), *Theory of Correspondences*, Wiley, New York.

Koopmans, Tjalling C. (1960), Stationary Ordinal Utility and Impatience, *Econometrica* 28, 287-309.

Koopmans, Tjalling C., Peter A. Diamond and Richard E. Williamson (1964), Stationary Utility and Time Perspective, *Econometrica* 32, 82-100.

Lucas, Robert E., Jr. (1978), Asset Prices in an Exchange Economy, *Econometrica* 46, 1429-1455.

Lucas, Robert E., Jr. and Nancy L. Stokey (1984), Optimal Growth with Many Consumers, *J. Econ. Theory* 32, 139-171.

Magill, Michael J. P. (1981), Infinite Horizon Programs, *Econometrica* 49, 679-711.

Magill, Michael J. P. and Kazuo Nishimura (1984), Impatience and Accumulation, *J. Math. Anal. Appl.* 98, 270-281.

Majumdar, Mukul (1975), Some Remarks on Optimal Growth with Intertemporally Dependent Preferences in the Neoclassical Model, *Rev. Econ. Stud.* 42, 147-153.

McKenzie, Lionel W. (1974), Turnpike Theorems with Technology and Welfare Function Variable, in *Mathematical Models in Economics* (J. Los and M. W. Los, eds.), North-Holland, New York.

Mirman, Leonard J. and Itzhak Zilcha (1975), On Optimal Growth Under Uncertainty, *J. Econ. Theory* 11, 329-339.

Mitra, Tapan (1979), On Optimal Economic Growth with Variable Discount Rates: Existence and Stability Results, *International Econ. Rev.* 20, 133-145.

Nairay, Alain (1985), Recoverability of Uzawa Utiity Functionals Under Asset Price Lognormality, *J. Econ. Dynamics Control* 9, 241-250.

Peleg, Bezalel and Menahem E. Yaari (1970), Markets with Countably Many Commodities, *Internat. Econ. Rev.* 11, 369-377.

Prescott, Edward C. and Rajnish Mehra (1980), Recursive Competitive Equilibrium: The Case of Homogeneous Households, *Econometrica* 48, 1365-1379.

Reed, Michael and Barry Simon (1972), *Methods of Modern Mathematical Physics I, Functional Analysis*, Academic Press, New York.

Smart, D. R. (1974), *Fixed Point Theorems*, Cambridge University Press, Cambridge.

Stein, Elias M. (1956), Note on Singular Integrals, *Proc.* AMS 8, 250-254.

Streufert, Peter A. (1985), Dynamic Allocation with Consistent Intergenerational Benevolence, IMSSS Technical Report #471, Stanford.

Streufert, Peter A. (1986), Abstract Dyanmic Programming, University of Wisconsin-Madison, SSRI Workshop Series #8619

Sutherland, W. R. S. (1970), On Optimal Development in a Multi-Sectoral Economy: the Discounted Case, *Rev. Econ. Stud.* 37, 585-588.

Uzawa, H. (1968), Time Preference, the Consumption Function, and Optimum Asset Holdings, in *Value, Capital and Growth: Papers in Honour of Sir John Hicks* (J.N. Wolfe, ed.), Edinburgh University Press, Edinburgh.

Waldman, K. H. (1985), On Bounds for Dynamic Programs, *Math. Operations Research* 10, 220-232.

Wessels, J. (1977), Markov Programming by Sucessive Approximations with Respect to Weighted Supremum Norms, *J. Math. Anal. Appl.* 58, 326-335.

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