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Symmetries, Equilibria and the Value Function

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SYMMETRIES, EQUILIBRIA

AND THE VALUE FUNCTION*

by

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Abstract

This paper presents a geometric approach (symmetries) to dynamic economic problems that integrates the solution procedure with the economics of the problem. Techniques for using symmetries are developed in the context of portfolio choice, optimal growth, and dynamic equilibria. Information on preferences, budget sets, and technology is combined to explicitly compute the solution. By focusing on the geometry of the underlying economic structure, the symmetry method can handle many types of problems with equal ease. Given an appropriate economic structure, it is immaterial whether the problem is in continuous or discrete time, is deterministic or stochastic with a Brownian, Poisson or other process, uses a finite or infinite time horizon, or even whether the rate of time preference is fixed or variable. These details are unimportant as long as the geometry is unchanged. All cases are treated in a unified manner. A major strength of the symmetry technique is its ability to ferret out the solutions to complex models with simple underlying economic structures. For example, a previously unsolved optimal growth model with both time-varying discount rates and technology is easily solved via symmetries.

1. Introduction

This paper presents a geometric approach (symmetries) to dynamic economic problems that integrates the solution procedure with the economics of the problem. Techniques for using symmetries are developed in the context of portfolio choice, optimal growth, and dynamic equilibria. Information on preferences, budget sets, and technology is combined to explicitly compute the solution. By focusing on the geometry of the underlying economic structure, the symmetry method can handle many types of problems with equal ease. Given an appropriate economic structure, it is immaterial whether the problem is in continuous or discrete time, is deterministic or stochastic with a Brownian, Poisson or other process, uses a finite or infinite time horizon, or even whether the rate of time preference is fixed or variable. These details are unimportant as long as the geometry is unchanged. All cases are treated in a unified manner.

Symmetries exhibit various interesting features. A major strength of the symmetry technique is its ability to ferret out the solutions to complex models with simple underlying economic structures. A previously unsolved optimal growth model with both time-varying discount rates and technology is easily solved via symmetries. Symmetries can also be used to transform problems into a simpler form, as will be demonstrated for hyperbolic absolute risk aversion (HARA) felicity. Symmetries are well-adapted for dealing with equilibrium problems. For example, they can demonstrate the uniqueness of equilibrium. Finally, the symmetries do not require the transversality condition, although it will hold if necessary for an optimum.

In contrast, most dynamic economic models are solved on a case-by-case basis. Although the methods of dynamic programming bring some order to the subject, they do not fully exploit the economic structure of the models.

Stochastic control problems are a case in point. They can rarely be explicitly solved even though the Bellman partial differential equation characterizes the solutions.

Explicit solutions have been found in a few cases involving portfolio selection and asset pricing.[1] Although Samuelson (1969) derived the solution for one problem, most of these solutions were obtained by the method of "divine revelation."[2] Guess the solution and plug it in. If it works, fine; if not, try another guess. This reliance on trial solutions is not totally satisfactory. The most general result was found by Danthine and Donaldson (1981).[3] Their theorem still leaves the solution technique unconnected with the economic structure of the problem. Why the problem has a solution of this form is left unexplained.

The rationale for the symmetry approach is most easily seen by considering a simple portfolio problem. It has two important characteristics—a linear homogeneous budget constraint and a homogeneous valuation functional defined over consumption paths. In ordinary demand theory, the combination of these two properties gives rise to a homogeneous indirect utility function.

One way to demonstrate this is the following geometric argument. Suppose there are only two goods and a linear budget constraint. An increase in income expands the budget set uniformly, as in the usual textbook diagram. When preferences are homogeneous (or even homothetic), this uniform expansion leaves the indifference map unchanged. At the optimum, the relative quantities of each good are the same, only the absolute amount changes. Plugging this in the utility function reveals that the indirect utility function is homogeneous in income. The situation is really no different in the portfolio problem.

This type of argument is not limited to uniform expansion of the feasible set. When there is production, increasing the endowment can cause a lop-sided expansion (dilation) of the feasible set. If dilation does not change the preference ordering, the expansion takes optima to optima. We can then easily calculate the indirect utility (value) function.

When there are more goods, the geometry may not look quite so simple. Rather than inspecting the shapes of budget sets and indifference maps on a diagram, we must use more powerful methods. Mathematically, these geometric relationships are expressed by using certain transformations (symmetries) that are based on the economic structure.[4] The symmetries generate generalized notions of homogeneity and homotheticity. The uniform expansion used to define homogenity is replaced by another transformation—the symmetry. This generalized homotheticity turns symmetries into powerful tools. With them, we can discover many of the economically important properties of the solutions. Their full power is most apparent when dealing with more complex problems. This is especially true in dynamic competitive equilibrium problems. The effects on many agents can easily be aggregated, even when the agents are heterogeneous.

Sections Two and Three introduce the symmetry technique. Section Two sets up the basic abstract framework, and presents a theorem relating symmetries to the value function. Section Three uses a simple linear symmetry to examine some portfolio problems of Merton (1969, 1971, 1973) and Fischer (1975). Two main conditions are used. The first is linear homogeneity of the state equation, as in typical budget constraints. The second condition is that the felicity function be either homogeneous or logarithmic. The constant relative risk aversion felicity functions obey this restriction.

This sets the stage for using symmetries as tools. Section Four shows how the symmetry approach works in more complex cases where the felicity function is neither homogeneous nor logarithmic. Symmetries apply to Merton's portfolio problem even for the more general hyperbolic absolute risk aversion felicity functions. The same arguments apply to more general stochastic processes. In particular, they apply for Poisson processes.

Section Five examines Ramsey problems with Cobb-Douglas production functions. Two types of model are considered. In the first, the symmetries determine the form of the value function for a Ramsey problem with discounting originally solved by Mirman and Zilcha (1975). The undiscounted version (Mirman and Zilcha, 1977) can be handled in exactly the same way. It makes no difference to the symmetries if the problem is discounted. The second example is a variant of the Mirman and Zilcha model inspired by Mitra (1979) that allows for non-stationary production and discount rates. Many time-varying models may be explicitly solved via the symmetry technique.

Section Six examines a sample equilibrium problem. This is Michener's (1982) version of Lucas' (1978) asset pricing model. In fact, the symmetry applies to the more general asset pricing model of Brock (1982). Even in this wider framework, they can show that Lucas' stationary asset pricing function gives all possible equilibrium prices. Interestingly, the transversality condition need not be invoked to prove uniqueness of the price sequence.

2. Symmetries and Their Properties

A form of Markov decision model provides a useful framework for introducing the symmetry concept. Let $z = (m,c,a,\pi)$ denote a Markov process over the probability space Ω with index set $\mathcal{T}$ that takes values in $\mathcal{M} \times \mathcal{C} \times \mathcal{A} \times \mathcal{P}$.[5] The index set can be either the interval $[0,T]$ (for continuous processes) or the set $\{0,\ldots,T\}$ (for discrete processes). Of course, T can be infinite. The Markov process z is assumed to obey a system of stochastic differential (or difference) equations $L(z) = 0$. These are the *evolution equations*.

The process z includes four types of variables. There is an endogenous state variable m that will usually be income or wealth. There are two types of action variables. The consumption-type variables c appear in the objective functional. Other actions that do not enter the objective are denoted by a. Asset demands fall in this category. Finally, π includes all exogenous parameters. These can be fixed, variable, or even stochastic.

Various restriction on the values state variables can take are summarized by the sets $\mathcal{M}$, $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{P}$. The economics of the problem will impose some structure on these sets. It may be important in the solution. If consumption must be non-negative, set $\mathcal{C} = \{c \in \mathbb{R}^n: c \geq 0\}$. When $\pi = \pi_0$ is fixed, take $\mathcal{P} = \{\pi_0\}$.

The objective functional defined over consumption paths will be denoted by V. Various types of objectives, such as recursive utility functions, are permitted.[6] Typically, the objective functional V will be additively separable with immediate reward (*felicity*) $u(c,s)$. In that case, $V(c,t) = \int_t^T u(c,s)\,ds$ for continuous processes, and $V(c,t) = \Sigma_{s=t}^T u(c,s)$ for discrete processes.

Let E_t denote the conditional expectation given $m(t)=m_t$.[7] The basic problem can be written:

$$J(m_t,t|\pi) = \sup \{E_t V(c,t) : L(z) = 0\}.$$

We will refer to J as the *value function* (indirect utility function). For simplicity, I write $J(m_t,t)$ if π is fixed, and $J(m)$ if t is also fixed.

To understand the geometric structure of the problem, focus on the budget set (feasible set). The *budget set* given initial state m_t and parameter π is $\mathbf{B}(m_t,t|\pi) = \{c \in \mathcal{C} : L(z) = 0 \text{ for some } z \text{ with } m(t) = m_t\}$.

The important question is how the budget set's geometry depends on the initial state m_t. Transformations that leave the evolution equations invariant will be used to investigate this. These transformations map the budget set to another budget set. Let $\mathbf{T} = T_1 \times T_2 \times T_3 \times T_4$ be an invertible (bijective) transformation from $\mathcal{M} \times \mathcal{C} \times \mathcal{A} \times \mathcal{P}$ to $\mathcal{M}' \times \mathcal{C}' \times \mathcal{A}' \times \mathcal{P}'$. It is a *symmetry* provided $L(\mathbf{T}z) = 0$ if and only if $L(z) = 0$.[8] Although the first problems I solve will use linear, time-independent symmetries, symmetries need not be linear. Sometimes the symmetries will be non-linear, sometimes they will be time-dependent, and sometimes they will even be stochastic. Later, I will even use symmetries that involve the exogenous parameters.

The following lemma shows the effect of a symmetry on the budget set. The mapping T_2 takes the budget set to another budget set with possibly different inital data and parameters. This is the key fact used in the Symmetry Theorem.

LEMMA. *If* $\mathbf{T}$ *is a symmetry, then* $\mathbf{B}(T_1 m_t | T_4 \pi) = T_2 \mathbf{B}(m_t | \pi)$.

PROOF. Let $c \in \mathbf{B}(m_t|\pi)$. Take z with $L(z) = 0$. Since $\mathbf{T}$ is a symmetry, $L(\mathbf{T}z) = 0$. Hence $T_2 c \in \mathbf{B}(T_1 m_t | T_4 \pi)$, so $T_2 \mathbf{B}(m_t|\pi) \subset \mathbf{B}(T_1 m_t|T_4\pi)$.

Since the T_i are invertible, we can apply the above result to $T'_i = T_i^{-1}$, $m'_t = T_1 m_t$ and $\pi' = T_4 \pi$ to get $T'_2 \mathbf{B}(m'_t|\pi') \subset \mathbf{B}(T'_1 m'_t|T'\pi')$. But $T'_1 m'_t = m_t$ and $T'_4 \pi' = \pi$, so $\mathbf{B}(T_1 m_t|T_4\pi) \subset T_2 \mathbf{B}(m_t|\pi)$. Hence $\mathbf{B}(T_1 m_t|T_4\pi) = T_2 \mathbf{B}(m_t|\pi)$. QED

SYMMETRY THEOREM. *If* $E_t V(T_2 c, t) = f_t[E_t V(c,t)]$ *for some family of increasing functions* f_t, *then* $J(T_1 m_t, t|T_4\pi) = f_t[J(m_t, t|\pi)]$.

PROOF.[9] Since

$$J(T_1 m_t, t|T_4\pi) = \sup \{E_t V(c,t) : c \in \mathbf{B}(T_1 m_t|T_4\pi)\}$$

we apply the lemma to get

$$\begin{aligned} J(T_1 m_t, t|T_4\pi) &= \sup \{E_t V(c,t) : c \in T_2 \mathbf{B}(m_t|\pi)\} \\ &= \sup \{E_t V(T_2 c, t) : c \in \mathbf{B}(m_t|\pi)\}. \\ &= \sup \{f_t[E_t V(c,t)] : c \in \mathbf{B}(m_t, t|\pi)\}. \end{aligned}$$

Now since f_t is increasing, we can pull it through the supremum to get

$$J(T_1 m_t, t|T_4\pi) = f_t[\sup \{E_t V(c,t) : c \in \mathbf{B}(m_t, t|\pi)\}]$$

So $J(T_1 m_t, t|T_4\pi) = f_t[J(m_t, t|\pi)]$. QED

Note that the family of functions f_t can be time-dependent. In fact, it can even be stochastic.

3. Simple Symmetries

When m is a scalar variable, and L is linear homogeneous in (m,c), the Symmetry Theorem determines the form of the solution for additively separable V when u(c,s) is either homogeneous or logarithmic in c.

COROLLARY 1. *Let m and c be real-valued and suppose V is an additively separable objective. Suppose* $L(\lambda m, \lambda c, a, \pi) = \lambda L(m,c,a,\pi)$, $u(\lambda c, s) = \lambda^\alpha u(c,s)$ *for* $\lambda > 0$, *and* $\mathcal{M}$ *and* $\mathcal{C}$ *are cones* ($\lambda\mathcal{M} = \mathcal{M}$, $\lambda\mathcal{C} = \mathcal{C}$). *If* $J(m_t, t|\pi)$ *exists, it has the form* $J(m_t, t|\pi) = A(t)\, m_t^\alpha$ *for some function* $A(t)$.

PROOF. Let $T_1 m = \lambda m$, $T_2 c = \lambda c$, $T_3 a = a$ and $T_4 \pi = \pi$ for $\lambda > 0$. Now $\lambda L(z) = L(Tz)$. Since these are simultaneously zero, **T** is a symmetry. Now, $V(T_2 c, t) = V(\lambda c, t) = \lambda^\alpha V(c,t)$ since $u(\lambda c, s) = \lambda^\alpha u(c,s)$. Set $f_t(V) = \lambda^\alpha V$. By the Symmetry Theorem,

$$J(\lambda m_t, t|\pi) = f[J(m_t, t|\pi)] = \lambda^\alpha J(m_t, t|\pi)$$

Now take $\lambda = 1/m_t$ and $A(t) = J(1, t|\pi)$ to get $J(m_t, t|\pi) = A(t)\, m_t^\alpha$. QED

COROLLARY 2. *Under the conditions of Corollary 1 with* $u(c,s) = f(s) \log c$, $J(m_t, t|\pi) = A(t) + F(t) \log m_t$, *where* $F(t) = \int_t^T f(s)\, ds$ *in continuous time and* $F(t) = \Sigma_{s=t}^T f(s)$ *in discrete time.*

PROOF. In continuous time, $V(\lambda c, t) = \int_t^T f(s) \log \lambda c \, ds = V(c,t) + F(t) \log \lambda$. Setting $f_t(V) = V + F(t) \log \lambda$, the Symmetry Theorem yields

$J(\lambda m_t, t|\pi) = F(t) \log \lambda + J(m_t, t|\pi)$. Now set $\lambda = 1/m_t$ and $A(t) = J(1, t|\pi)$. This shows $J(m_t, t|\pi) = A(t) + F(t) \log m_t$. Discrete time is similar. QED

An easy application is to a version of Merton's (1969, 1971) portfolio problem. In its simplest form, there are two assets. The safe asset pays a certain return r. The risky asset pays an expected return ρ and a stochastic return with variance σ, described by a Brownian motion z. The total return from a unit of the risky asset in a time interval dt is $[\rho\ dt + \sigma\ dz]$. Total wealth is m and a is the share of wealth held in the risky asset. Income from the safe asset is $[(1-a)mr\ dt]$ while income from the risky asset is $[am\rho\ dt + a\sigma\ dz]$. Since the consumption flow is c, the budget constraint can be written as the stochastic differential equation $L(m,c,a,\pi) = dm - a(\rho-r)m\ dt - (rm-c)\ dt - am\sigma\ dz = 0$ where $\pi = (r,\rho,\sigma)$. Of course, the solutions will be Markov processes (see Arnold, 1974). When short selling ($a < 0$ or $a > 1$) is permitted, the problem is

$$J(m_t, t) = \sup_{\{c,a\}} E_t \int_t^T c^{\alpha} e^{-\beta s}\, ds$$

$$\text{s.t.} \quad dm = L(m,c,a,\pi);$$

$$c \geq 0,\ m \geq 0; \quad m(t) = m_t.$$

Here $\mathcal{M} = \{m : m \geq 0\}$, $\mathcal{C} = \{c : c \geq 0\}$ and $\mathcal{A} = \{a\}$. If short selling were prohibited, take $\mathcal{A}' = \{a : 0 \leq a \leq 1\}$. In either case, Corollary 1 clearly applies. When the value function exists, it is given by $J(m_t, t) = A(t)\ m_t^{\alpha}$. Knowing the form of J in advance is quite useful. The Bellman equation can have extraneous solutions. Merton (1969) used a transversality condition to eliminate them. The symmetry approach automatically eliminates the extraneous solutions from consideration.

Many of the economically important results do not need the explicit solution for J, only the form determined by the corollaries. We can immediately see that the value function has absolute risk aversion $(1-\alpha)/m$ and relative risk aversion $(1-\alpha)$, regardless of what $A(t)$ is. This can be used in the first order conditions to show that consumption is linear in wealth and that the optimal asset share is $a = (\rho-r)/\sigma^2(1-\alpha)$. If this is in $\mathscr{A}'$, this is also the solution to the problem without short selling. Merton (1969, 1971, 1973) has computed $A(t)$ in detail. When short selling is prohibited, and $(\rho-r)/\sigma^2(1-\alpha)$ is not in $\mathscr{A}'$, all wealth must be held in only one of the assets ($a = 0$ or 1).

The same symmetry works on closely related versions of this problem. Logarithmic felicity functions can be treated the same way by using Corollary 2 in place of Corollary 1. These results can also be extended to the case where there are many risky assets. One such model, involving two state equations, is due to Fischer (1975). He uses a variant of the Merton model where there are two risky assets with returns $[\rho_1\, dt + \sigma_1\, dz_1]$ and $[\rho_2\, dt + \sigma_2\, dz_2]$. The price of the consumption good is given by a stochastic inflation process p with expected inflation rate π and variance σ_p. The consumer's wealth shares of the two risky assets are a_1 and a_2. This leaves $(1 - a_1 - a_2)$ as the wealth share of the safe asset. Two distinct symmetries arise from the economic structure of the problem. By using both, I can extract more information than would otherwise be possible.

Fischer's problem is

$$J(m_0,P_0) = \sup_{\{c,a_i\}} E_0 \int_0^\infty u(c,s)\, ds$$

$$\text{s.t.}\quad dm = [a_1(\rho_1-r) + a_2(\rho_2-r)]m\, dt + (rm-pc)\, dt + m(a_1\sigma_1\, dz_1 + a_2\sigma_2\, dz_2)$$

$$dp = \pi p\, dt + p\sigma_p\, dz_p$$

$$m \geq 0,\ c \geq 0,\ m(0) = m_0,\ p(0) = p_0.$$

There is a second symmetry acting on the price level p in addition to the familiar symmetry $T_1 m = \lambda m$, $T_2 c = \lambda c$ and $T_3 a = a$. The second symmetry is $S_1(m,p) = (\lambda m, \lambda p)$, $S_2 c = c$ and $S_3 a = a$. This symmetry is particularly interesting since it leaves the budget set unchanged. It expresses the fact that the budget set depends only on real wealth, not nominal wealth. By the Symmetry Theorem, applied to $\mathbf{S} = (S_1, S_2, S_3)$, $J(\lambda m_o, \lambda p_o) = J(m_o, p_o)$. Setting $\lambda = 1/p_o$ yields $J(m_o, p_o) = J(m_o/p_o, 1) = I(m_o/p_o)$ for some function I. The value function J is also a function of real, not nominal, wealth.

We can use this to derive Fischer's important relation

$$J_{mp}p/J_{mm}m = -(J_m/J_{mm}m) - 1$$

(equation 20 in Fischer). Although Fischer uses this equation, he never actually shows it is true. Rather, he asserts that consumption depends on real wealth and uses a first order condition to get his equation (20). We can go the opposite way to see that Fischer's assertion is true. Use the first order condition $u_c(c,s) = pJ_m$. As $pJ_m = p(I'(m/p)/p) = I'(m/p)$, and since consumption satisfies $u_c(c,s) = I'(m/p)$, it is a function of real wealth.

In the special case where felicity is homogeneous (or logarithmic) we can use the standard symmetry $\mathbf{T}$ and the two corollaries to get, respectively, $J(m_o, p_o) = A(m_o/p_o)^{\alpha}$ and $J(m_o, p_o) = A + [\log (m_o/p_o)] \int_0^{\infty} f(s)\, ds$.

In these simple cases, T_3 and T_4 were identity transformations. This is true for most known linear examples. The same type of symmetry can be applied to asset pricing models with multiple consumption variables.[10] The more complex models in the following sections will have non-trivial T_3 and T_4.

The key fact behind the corollaries is that the symmetry $\mathbf{T}$ is related to the objective in a simple way: $V(T_2 c, t) = V(\lambda c, t) = \lambda^{\alpha} V(c,t)$ or $V(T_2 c, t) = \log \lambda + V(c,t)$. Maxima are transformed into maxima. The optimal controls (c^*, a^*) for the problem with initial conditions $m(t) = m_t$, are transformed into the optimal controls $(T_2 c^*, T_3 a^*)$ for the problem with initial conditions m(t)

$= T_1 m_t$. When the optimal controls exist, this fact can be quite useful. Hahn (1970) used it to determine optimal saving and consumption.

Symmetries can also help illuminate existing results. Danthine and Donaldson (1981) have shown that consumption is a not a linear function of output in stochastic control problems with Cobb-Douglas production and homogeneous logarithmic felicity, although it is linear for logarithmic felicity. A close look at the appropriate symmetry explains why. In fact, this result appears even in a two-period deterministic setting. Neither an infinite-horizon nor uncertainty are necessary.

Consider Cobb-Douglas production, $c_2 = (m_0 - c_1)^{1/2}$. As m_0 is increased to λm_0, the production frontier expands in a lop-sided manner from the parabola AA to the parabola BB in Figure 1. This expansion is performed by the symmetry $T_2(c_1, c_2) = (\lambda c_1, \lambda^{1/2} c_2)$. A given optimum S is mapped to T by this symmetry. The indifference maps of most constant elasticity of substitution (CES) utility functions are distorted by this transformation. For example, the Leontieff utility function $u_1(c_1, c_2) = \min\,(c_1, c_2)$ is transformed into $u_2(c_1, _2) = \min\,(\lambda c_1, \lambda^{1/2} c_2)$. Clearly, the indifference curves are different. Only the Cobb-Douglas indifference curves are undistorted, and so only Cobb-Douglas utility has T as the new optimum. In the Leontieff case, the both the old optimum S and the new optimum R lie on the 45° line. For general CES utility, the actual location of the new optimum depends on the elasticity of substitution σ. When S is the original optimum, the new optimum will be R when $\sigma = 0$ (Leontieff), between R and T for σ between 0 and 1, T for $\sigma = 1$ (Cobb-Douglas), and to the right of T for σ greater than 1. In finite-horizon discounted models, homogeneous felicity yields a CES utility function, while logarithmic felicity gives Cobb-Douglas utility. As in the two-period model, only the logarithmic case is well-behaved.

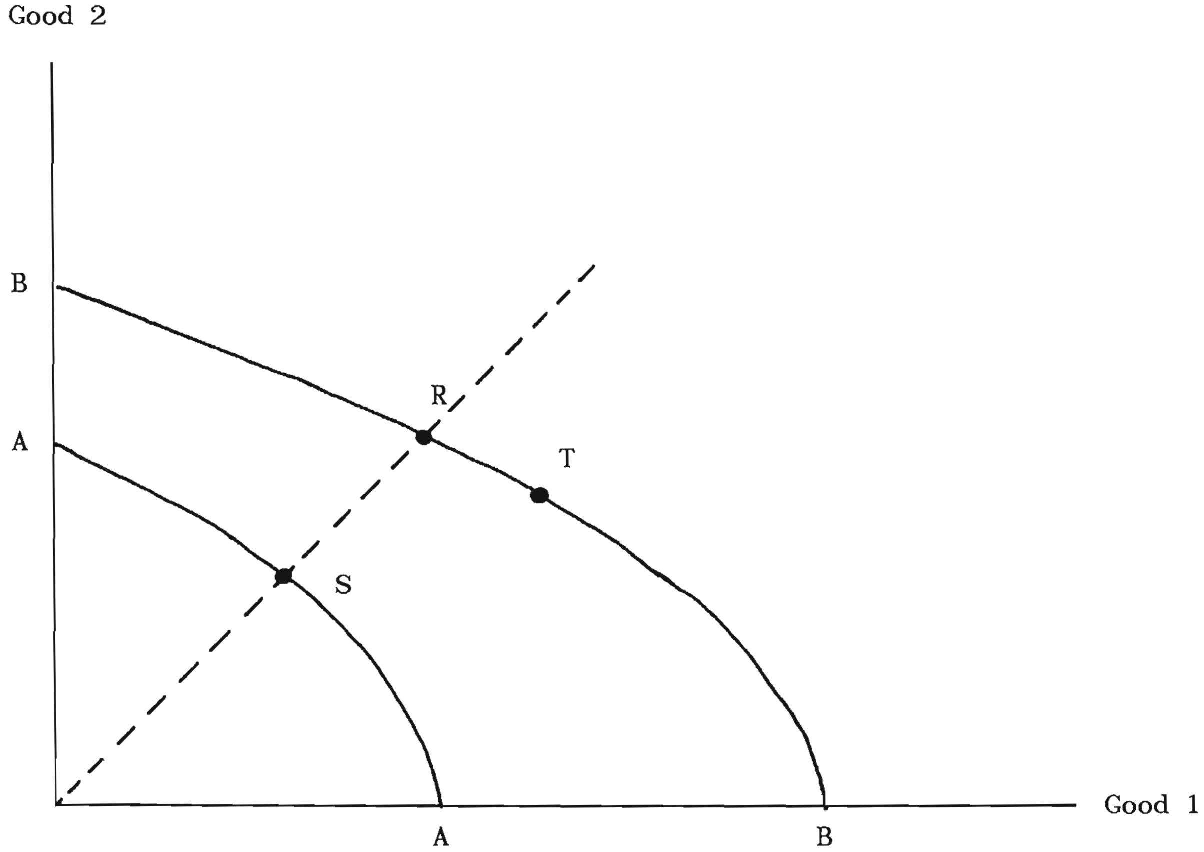

Figure 1

4. Symmetries With HARA Felicity

Another way to use symmetries is to transform a problem into an easier-to-solve form. This works on Merton's (1971, 1973) portfolio problem for the HARA class function $u(c,s) = (c + \eta)^{\alpha} e^{-\beta s}/\alpha$ where $c + \eta \geq 0$, $\alpha < 1$ and α, $\eta \neq 0$. The constraint on c, $c + \eta \geq 0$, is a bit different from the homogeneous Merton problem. The strategy is simple. Use a symmetry with $T_2 c = c + \eta$ to turn this into a homogeneous felicity function.

This sort of symmetry is a bit different from the previous examples because the constraint changes. The solution is to use a symmetry with $(T_1, T_2, T_3) : \mathcal{M} \times \mathcal{C} \times \mathcal{A} \to \mathcal{M}' \times \mathcal{C}' \times \mathcal{A}'$. The T_i must still be invertible and satisfy the budget constraint

$$dm = a(\rho - r)m\ dt + (rm - c)\ dt + am\sigma\ dz. \tag{1}$$

Let $m^* = T_1 m$ and $a^* = T_3 a$. For (T_1, T_2, T_3) to be a symmetry, the transformed variables must solve

$$dm^* = a^*(\rho - r)m^*\ dt + (rm^* - c - \eta)\ dt + a^* m^* \sigma\ dz.$$

Substituting for c from (1) gives us

$$dm^* = dm + (a^* m^* - am)[(\rho - r)\ dt + \sigma\ dz] + r(m^* - m)dt - \eta\ dt$$

Choose a^* to eliminate the stochastic part of the equation. Letting $a^* = a(m/m^*)$ so that $a^* m^* = am$ yields

$$d(m^* - m) = r(m^* - m)\ dt - \eta\ dt. \tag{2}$$

This is a crucial point. Equation (2) has many solutions. Which one should we choose? Since $\mathcal{M}' = \{m^*: m^* \geq 0\}$, our choice determines the constraint set $\mathcal{M}$ for the inhomogeneous problem. One reasonable possibility is to require that terminal wealth m(T) be non-negative. Under this condition, any borrowing must be repaid by time T. The boundary condition $m^*(T) = m(T)$ will insure this. This leads to Merton's solution. A different choice here would mean a different constraint set $\mathcal{M}$.

With $m^*(T) = m(T)$ as boundary condition, the solution is

$$T_1 m = m^* = m + (\eta/r)\ [1 - e^{-r(T-t)}].$$

This gives $\mathcal{M} = \{m\colon m + (\eta/r)\ [1 - e^{-r(T-t)}] \geq 0\}$.

For $t < T$ and $\eta > 0$ there is a positive probability of negative wealth, but the asset holder must end up with $m(T) \geq 0$.[11] Of course, the value function is

$$J(m_t, t) = A(t) \left[m_t + (\eta/r)\ [1 - e^{-r(T-t)}] \right]^{\alpha}.$$

Another choice of constraint would give a different solution. The economic structure of the problem enters in a decisive way through this constraint.

In addition to showing the importance of paying close attention to the constraints, this symmetry is different in other respects. It is affine, not linear, and the transformation is time-dependent. This last feature appears again with non-linear production functions.

5. Symmetries and Non-Linear Production

Another application of symmetries is to models involving Cobb-Douglas production and logarithmic felicity. In this section, I use symmetries on a stationary Ramsey problem (Mirman and Zilcha) and a non-stationary version of the same problem inspired by Mitra. The same method will also apply to more complex optimal growth models of this type, as those used in Radner (1966) and Long and Plosser (1983).

5.1 *Time Stationary Felicity*

A simple stochastic Ramsey problem, due to Mirman and Zilcha (1975), is

$$J(y) = \sup_{c_t} E_0 \left[\Sigma_{t=1}^{\infty} \delta^{t-1} \log c_t \right]$$

$$\text{s.t.}\quad c_t + k_t = k_{t-1}^{\rho_t};\quad t \geq 2 \tag{3}$$

$$c_t \geq 0,\ k_t \geq 0;\quad c_1 + k_1 = y.$$

where $\delta < 1$ is the discount factor and the ρ_t are independent random variables with mean $\rho < 1$.

For a symmetry to be useful, it must preserve the preference ordering. With logarithmic felicity, that means that $T_2 c_t = \alpha_t c_t^{\theta}$ for some α_t and θ. Let $T_1 k_t = k_t^*$ and $T_2 c_t = c_t^*$ be the transformation associated with $y^* = \lambda y$. Equation (3) must be satisfied by both the transformed and untransformed variables, so when $c_t = 0$, $k_t = k_{t-1}^{\rho_{t-1}}$ and $k_t^* = (k_{t-1}^*)^{\rho_{t-1}}$. Setting $z_t = k_t^*/k_t$, we see that z_t satisfies $z_t = z_{t-1}^{\rho_{t-1}}$ with $z_1 = \lambda$. This has solution $z_t = \lambda_t$ where $\lambda_t = \lambda^{(\prod_{i=1}^{t-1} \rho_i)}$. Notice that although ρ_t is stochastic, the transformation can be defined as easily as if it were deterministic.

Substituting the expressions for k_t^* and c_t^* into (3), we see that this is indeed a symmetry provided $\alpha_t = \lambda_t$ and $\theta = 1$. Applying the discrete-time analog of Theorem 1 shows that

$$J(\lambda y) = E_0 \left[\Sigma_{t=1}^{\infty} \delta^{t-1} (\textstyle\prod_{i=0}^{t-1} \rho_i) \log \lambda \right] + J(y).$$

Setting $\lambda = 1/y$ and using the fact that the ρ_t are independent with mean ρ, we get

$$J(y) = J(1) + [1/(1 - \rho\delta)] \log y.$$

The symmetries also show that the optimal policy function is linear in production y.

The same symmetry can be used on the undiscounted version of this problem (Mirman and Zilcha, 1977) to get

$$J(y) = J(1) + (1 - \rho)^{-1} \log y.$$

In general $J(1) \neq 0$, although Mirman and Zilcha erroneously give $J(1) = 0$. Without discounting, $J(1)$ is determined by normalizing J so that the Golden Rule has value zero. When ρ_t is deterministic ($\rho_t = \rho$), the Golden Rule initial endowment is $g = \rho^{\rho/(1-\rho)}$ and $J(1) = -(1 - \rho)^{-1} \log g$. Therefore, $J(y) = (1 - \rho)^{-1} \log (y/g)$.

5.2 *Non-Stationary Felicity*

An interesting variant on this problem was studied by Mitra (1979). Using a specialization of McKenzie's (1974) general time-varying model, Mitra considered a fixed felicity function with a time-varying discount rate. A further generalization, which I will examine, is to allow the technology to vary with time as well. For notational convenience, I only consider the deterministic case. These results may be extended to admit stochastic production as well. To insure that symmetries apply, I will consider Cobb-Douglas production and logarithmic felicity as above. Let $\gamma = (\gamma_1, \gamma_2, \ldots)$, $\rho = (\rho_1, \rho_2, \ldots)$, and $\Delta = (\delta_1, \delta_2, \ldots)$ be given. The technology at time t is given by $f_t(x_t) = \gamma_t x_t^{\rho_t}$ with δ_t representing the discount factor at time t. The Ramsey problem $P(y, \Delta, \rho, \gamma)$ becomes:

$$J(y|\Delta,\rho,\gamma) = \sup \Sigma_{t=1}^{\infty} \delta_t \log c_t$$

$$\text{s.t.} \quad c_t + x_t = \gamma_{t-1} x_{t-1}^{\rho_{t-1}} \quad \text{for } t>1$$

$$c_1 + x_1 = y; \quad c_t, x_t \geq 0$$

Two symmetries combine to find the value function. The first transforms this into a problem with $\gamma_t = 1$. The symmetry $\mathbf{S}(c_t, x_t) = \exp \sigma_t \, (c_t, x_t)$ does this when σ_t solves the difference equation $\sigma_t = \log \gamma_{t-1} + \rho_{t-1}\sigma_{t-1}$ with $\sigma_1 = 0$. The symmetry used on the Mirman-Zilcha model then does the rest.

To illustrate this, consider the case $\delta_t = \delta^{t-1}$, $\rho_t = \rho$ and $\gamma_t = e^{\gamma t}$. For the first symmetry, $\sigma_t = \gamma(t-1) + \rho\sigma_{t-1}$ and $\sigma_1 = 0$. As is easily verified, the solution to this difference equation is $\sigma_t = \gamma(\rho^t - \rho t + t - 1)(1-\rho)^{-2}$. The value function then obeys $J(y|\delta,\rho,\gamma) = J(y|\delta,\rho,0) + \gamma\delta/[(1-\delta)^2(1-\rho\delta)]$.

This transforms the problem into a deterministic Mirman-Zilcha model. Its solution yields $J(y|\delta,\rho,\gamma) = A + \gamma\delta/[(1-\delta)^2(1-\rho\delta)] + (1-\rho\delta)^{-1}\log y$ where A is constant. An interesting point about this expression is that it permits computation of the optimal policy function, and hence facilitates explicit computation of the optimal path. For this, use the Bellman equation

$$J(y|\delta,\rho,\gamma) = \sup \{\log c + \delta J(e^{\gamma}(y-c)^{\rho}|\delta,\rho,\gamma)\}.$$

Plugging in the expression for the value function, and using the first order condition, shows that the optimal choice of c_1 is $(1-\rho\delta)y$. Similarly, $c_t = (1-\rho\delta)y_t$ where y_t is the income available at time t. The optimal consumption path is then $c_t = \bar{c}\exp\{\gamma(\rho^t - \rho t + t - 1)(1-\rho)^{-2} + \rho^{t-1}\log(y/g)\}$ where $\bar{c} = (1-\rho\delta)g$ and $g = (\rho\delta)^{\rho/(1-\rho)}$ denote steady state consumption and income, respectively.

Although technology grows at a constant rate, the growth rate of consumption varies. Asymptotically, the optimal path grows at rate $\gamma/(1-\rho)$. At finite times, its behavior depends on the sign of $[\log(y/g) + \gamma\rho/(1-\rho)^2]$. Consumption grows at an increasing, constant or decreasing rate as $[\log(y/g) + \gamma\rho/(1-\rho)^2]$ is negative, zero or positive. In the first two cases, consumption is monotonically increasing in time. The last case involves another interesting possibility. With a large initial capital stock, the optimal consumption can be U-shaped.[12]

Examination of the more general case $0 < \rho_t = \rho < 1$ with δ_t and γ_t arbitrary is also interesting. By the same procedure,

$$x_t^* = (1 - \delta_t/\mu_t)(1 - \delta_{t-1}/\mu_{t-1})^{\rho}\ldots(1 - \delta_1/\mu_1)^{\rho^{t-1}} y^{\rho^{t-1}}$$

Where $\mu_j = \Sigma_{t=j}^{\infty} \delta_t \rho^{t-j}$. Note that μ_j must be finite for all j if the problem is to be well-posed. Let x^* have initial stock x and z^* have initial stock z. Since $0 < (1 - \delta_t/\mu_t) < 1$,

$$|x_t^* - z_t^*| \leq |x^{\rho^{t-1}} - z^{\rho^{t-1}}| \to 0.$$

The optimal paths from different capital stocks are asymptotic to each other. This is known as the twisted turnpike. Mitra established this for the generic stationary technology case (1979). With time-varying production, the twisted turnpike need not hold.[13]

6. Equilibrium Models

The symmetries of Section Five can be applied to equilibrium models with Cobb-Douglas production and logarithmic felicity. In such a setting, the same symmetry can be applied to all households. Provided a steady state is known, other equilibria may be investigated. One relatively simple equilibrium model is Michener's (1982) version of Lucas' (1978) asset pricing model. In addition to the value function, we also need to find the equilibrium pricing function. This causes Michener some mild embarassment. Although Lucas did show that bounded utility functions have a unique equilibrium pricing function, his uniqueness theorem does not apply here. Fortunately, Michener's answer is unique, and the symmetries can show it. Furthermore, even if we allow non-stationary prices à la Brock (1982; Malliaris and Brock, 1982), equilibrium prices are still unique. Michener's problem must be broken into two parts. The first is solved conditional on asset prices. The second determines the equilibrium prices. The first problem is

$$V(y,z|p) = \sup_{\{c,x\}} E_0 \left[\Sigma_{t=1}^{\infty} \delta^{t-1} \log c_t \right]$$

$$\text{s.t.} \quad \log y_{t+1} = \rho \log y_t + \epsilon_t$$

$$c_t + p_t x_t = y_t z_t + p_t z_t; \quad z_{t+1} = x_t$$

$$z_t \geq 0, \quad c_t \geq 0; \quad z_1 = z, \quad y_1 = y.$$

Of course $0 \leq \rho < 1$, $0 < \delta < 1$ and $\epsilon_t \sim N(0,\sigma^2)$.

In each time period the consumer chooses x_t, the amount of the asset to hold at the end of the period, and consumption c_t. The consumer's asset holding is carried over and becomes the initial asset holding z_{t+1} for the next period.

This problem, like Fischer's, has two sets of symmetries. The first, based on the linear budget constraint $c_t + p_t x_t = y_t z_t + p_t z_t$, is $T_1(y_t,z_t) = (y_t,\lambda z_t)$, $T_2 c_t = \lambda c_t$, and $T_3 x_t = \lambda x_t$. By Corollary 2,

$$V(y,\lambda z|p) = (1-\delta)^{-1} \log \lambda + V(y,z|p).$$

Hence $V(y,z|p) = V(y,1|p) + (1-\delta)^{-1} \log z$.

Further, if x_t^* is optimal from z, λx_t^* is optimal from λz at the same prices. Hence, p_t is not only an equilibrium price sequence for z, so that $x_t^* = z$, but is also an equilibrium price sequence for λz. Equilibrium prices are unaffected by changes in initial wealth z.

The second symmetry is a bit different since it also involves the price sequence. This symmetry is based on the Cobb-Douglas technology $\log y_{t+1} = \rho \log y_t + \epsilon_t$, and is a deterministic version of the Mirman-Zilcha symmetry. It is $S_1(y_t,z_t) = (\lambda_t y_t,z_t)$, $S_2 c_t = \lambda_t c_t$, $S_3 x_t = x_t$, and $S_4 p_t = \lambda_t p_t$ where $\lambda_t = \lambda^{\rho^t}$. This symmetry maps equilibrium prices into equilibrium prices since x_t is unchanged by it.

PROPOSITION. *For each initial y, there is a unique equilibrium price sequence. It is given by the pricing function* $p(y_t) = \delta(1-\delta)^{-1} y_t$.

PROOF. An application of the Principle of Optimality shows that $q_t = p_{t+1}$ are equilibrium prices for (y_2, z). Another application yields

$$V(y,z|p) = \sup_{x_1} \{\log [yz + p_1(z - x_1)] + \delta\, E_0\, V(y_2, x_1|q)\}.$$

Now **T** shows that q_t are equilibrium prices for (y_2, z') for any z'. This symmetry also tells us $V(y_2, x_1|q) = V(y_2, 1|q) + (1-\delta)^{-1} \log x_1$. Armed with this information, we can now apply the first order conditions at $x_1 = z$ and find $p_1 = \delta(1-\delta)^{-1}\, y$.

Of course, a similar argument can be applied to q_t. Hence, $p_2 = \delta(1-\delta)^{-1}$ y_2 also. A simple induction shows that $p_t = \delta(1-\delta)^{-1}\, y_t$. This is necessarily unique. QED

The point is that the only equilibrium prices are actually given by the stationary equilibrium pricing function $p(y) = \delta(1-\delta)^{-1}\, y$. Now let $V(y,z) = V(y,z|p_t(y))$ be the equilibrium value function. Using the Symmetry Theorem on **S** shows $V(\lambda y, z) = (1-\rho\delta)^{-1} \log \lambda + V(y,z)$. Combining this with the results using **T** gives

$$V(y,z) = A + (1-\delta)^{-1} \log z + (1-\rho\delta)^{-1} \log y$$

for some constant A.

As is usual with the symmetry technique, the actual nature of the stochastic term was irrelevant. We will get the same results whenever this problem is well-posed.

7. Conclusion

This paper has shown how symmetry arguments can be used to solve various kinds of maximization problems. In addition to the examples presented here, symmetries are able to solve various other problems. They can also be used on the exponential utility functions used by Holmstrom and Milgrom (1986) and Chang (1986). The type of symmetry used on the Mirman-Zilcha example can be applied to more general problems of the same type, such as Radner (1966) or Long and Plosser (1983). Precott and Mehra's (1980; Mehra, 1984) recursive competitive equilibrium is another model where symmetries can be helpful.

Other types of equilibrium models may be examined. Becker's (1980) Ramsey equilibrium, where agents may not borrow against future wage income is such an example (Boyd, 1986a).[14] One interesting application of this is found by adding capital taxation to the model. With appropriate preferences and technology, symmetries can be used to calculate the transition path between steady states when tax rates are changed (Boyd, 1986b). The symmetries give us an expression for each individual's utility, and can be used to analyze welfare.

Elementary techniques for finding symmetries were presented in the examples. In many cases, the question of existence of symmetries remains open. The Noether theorem should prove helpful. It plays a key role in investigations of a related type of invariance in Sato (1981), Sato and Nono (1983) and Logan (1977).

Footnotes

1. Typical cases include Merton (1969, 1971), Samuelson (1969), Gertler and Grinols (1982), Malliaris and Brock (1982) and Michener (1982).

2. The major exceptions are Hahn (1970) and Mirrlees (1974) who use an embryonic form of the symmetry technique.

3. A general result for homogeneous felicity without uncertainty, is due to Mino (1983). His proof involved manipulation of the Hamiltonian.

4. This is the same type of symmetry concept used in physics. See Weyl (1952).

5. Each of the sets involved is a subset of some vector space.

6. See Koopmans (1960) for details on recursive preferences.

7. Note that the conditional expectation is usually a random variable. For it to be constant, m_t must be constant.

8. More generally, **T** is a symmetry between problems with evolution operators L and L′ provided $L(\mathbf{T}z) = 0$ if and only if $L'(z) = 0$. This expanded notion of symmetry will prove useful in Section Five.

9. When a maximum exists, it can be used to simplify the proof.

10. Gertler and Grinols (1982) use a money-in-the-felicity-function model where the felicity function is $u(c_1,c_2) = A \log c_1 + B \log c_2$ with c_1 denoting consumption and c_2 being real money balances.

11. The point is that $\log m^*$ is a Brownian motion. Thus there is a positive probability of $\log m^* < \log \{(\eta/r) [1 - e^{-r(T-t)}]\}$ for $t < T$, and m may be negative. Further, as $c^* = \theta m^*$ for some constant θ, there is also a positive probability of negative consumption. Further interpretation is required to make sense of this case. Merton works in a partial equilibrium setting, and c need only represent consumption out of wealth. Thus each agent may receive η units of the consumption good in each period, independent of wealth yielding total consumption $c + \eta$. An alternative is to require wealth and consumption be non-negative. This is the path followed by Sethi and Taksar (1986).

12. For example, with $\gamma = \rho = ½$, $0 < \delta < 1$ and $y = g^2 = \delta^2/4$, $c_t = \bar{c} \exp \{t + 3(½)^{t-1}\}$. Consumption in the first three periods is $c_1 = \bar{c}e^4$, $c_2 = \bar{c}e^{3.5}$ and $c_3 = \bar{c}e^{3.75}$.

13. In the exogenous technical progress example, the twisted turnpike does not hold. This may be verified by applying l'Hôpital's rule to the explicit expression for $x_t = \rho\delta c_t/(1-\rho\delta)$. A modified turnpike result does hold since $|x_t - z_t|e^{-\gamma t/(1-\rho)} \to 0$.

14. The Ramsey equilibirium is further developed in Becker and Foias (1986) and Becker, Boyd and Foias (1986).

References

Arnold, Ludwig (1974), *Stochastic Differential Equations: Theory and Applications*, Wiley, New York.

Becker, Robert A. (1980), On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households, *Quart. J. Econ.* 95, 375-382.

Becker, Robert A., John H. Boyd III and Ciprian Foias (1986), The Existence of Ramsey Equilbirium, Working Paper, Indiana University.

Becker, Robert A. and Ciprian Foias (1986), A Characterization of Ramsey Equilibrium, *J. Econ. Theory*, forthcoming.

Boyd, John H., III (1986a), *Preferences, Technology and Dynamic Equilibria*, PhD. Dissertation, Indiana University.

Boyd, John H., III (1986b), Dynamic Tax Incidence with Heterogeneous Households, Working Paper, University of Rochester.

Brock, William A. (1982), Asset Prices in a Production Economy, in *The Economics of Information and Uncertainty* (J. J. McCall, ed.), University of Chicago Press, Chicago.

Chang, Fwu-Ranq (1986), A Theory of the Consumption Function: The Case of Noncapital Income Fluctuations, Working Paper, Indiana University.

Danthine, J.-P. and John B. Donaldson (1981), Stochastic Properties of Fast vs. Slow Growing Economies, *Econometrica* 49, 1007-1033.

Fischer, Stanley (1975), The Demand for Index Bonds, *J. Pol. Econ.* 83, 509-534.

Gertler, Mark and Earl Grinols (1982), Monetary Randomness and Investment, *J. Mon. Econ.* 10, 239-258.

Hahn, Frank H. (1970), Savings and Uncertainty, *Rev. Econ. Stud.* 37, 21-24.

Holmstrom, Bengt and Paul Milgrom (1986), Aggregation and Linearity in the Provision of Intertemporal Incentives, Working Paper, Yale University.

Koopmans, Tjalling C. (1960), Stationary Ordinal Utility and Impatience, *Econometrica* 28, 287-309.

Logan, John D. (1977), *Invariant Variational Principles*, Academic, New York.

Long, John B., Jr. and Charles I. Plosser (1983), Real Business Cycles, *J. Pol. Econ.* 91, 39-69.

Lucas, Robert E., Jr. (1978), Asset Prices in an Exchange Economy, *Econometrica* 46, 1429-1455.

Malliaris, A. G. and William A. Brock (1982), *Stochastic Methods in Economics and Finance*, North-Holland, Amsterdam.

McKenzie, Lionel W. (1974), Turnpike Theorems with Technology and Welfare Function Variable, in *Mathematical Models in Economics* (J. Los and M. W. Los, eds.), North-Holland, New York.

Mehra, Rajnish (1984), Recursive Competitive Equilibrium: A Parametric Example, *Econ. Letters* 16, 273-278.

Merton, Robert C. (1969), Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case, *Rev. Econ. Stat.* 51, 247-257.

Merton, Robert C. (1971), Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *J. Econ. Theory* 3, 373-413.

Merton, Robert C. (1973), Erratum, *J. Econ. Theory* 6, 213-214.

Michener, Ronald W. (1982), Variance Bounds in a Simple Model of Asset Pricing, *J. Pol. Econ.* 90, 166-175.

Mino, Kazuo (1983), On the Homogeneity of Value Function of the Optimal Control Problem, *Econ. Letters* 11, 149-154.

Mirman, Leonard J. and Itzhak Zilcha (1975), On Optimal Growth Under Uncertainty, *J. Econ. Theory* 11, 329-339.

Mirman, Leonard J. and Itzhak Zilcha (1977), Characterizing Optimal Policies in a One-Sector Model of Economic Growth Under Uncertainty, *J. Econ. Theory* 14, 389-401.

Mirrlees, James A. (1974), Optimum Accumulation Under Uncertainty: The Case of Stationary Returns to Investment, in *Allocation Under Uncertainty* (Jacques H. Dreze, ed.), Wiley, New York.

Mitra, Tapan (1979), On Optimal Economic Growth with Variable Discount Rates: Existence and Stability Results, *International Econ. Rev.* 20, 133-145.

Prescott, Edward C. and Rajnish Mehra (1980), Recursive Competitive Equilibrium: The Case of Homogeneous Households, *Econometrica* 48, 1365-1379.

Radner, Roy (1966), Optimal Growth in a Linear-Logarithmic Economy, *International Econ. Rev.* 7, 1-33.

Samuelson, Paul A. (1969), Lifetime Portfolio Selection by Stochastic Dynamic Programming, *Rev. Econ. Stat.* 51, 239-246.

Sato, Ryzuo (1981), *Theory of Technical Change and Economic Invariance: Application of Lie Groups*, Academic, New York.

Sato, Ryzuo and Takayuki Nono (1983), *Invariance Principles and the Structure of Technology*, Springer, New York.

Sethi, Suresh P. and Michael Taksar (1986), A Note on Merton's "Optimum Consumption and Portfolio Rules in a Continuous-Time Model", Working Paper, University of Toronto.

Weyl, Hermann (1952), *Symmetry*, Princeton University Press, Princeton.

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