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by

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ABSTRACT

Models with a continuum of consumers and locations such that consumers can purchase goods in only one location are examined. Examples satisfying the usual assumptions but without equilibrium are given. An approximation by economies with a finite number of consumers is shown to fail. The results are related to the literatures concerning product differentiation and spatial economies.

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I. Introduction

Models with a continuum of agents, a continuum of goods, or both, have been used by economists for some time. Examples are Bewley (1972), Hildenbrand (1974), Mas-Colell (1975) and Jones (1983,1984), although this list does not contain a fraction of the literature. The reasons for using such models include mathematical ease and the means to make precise the hypotheses of perfect competition. Models with a continuum of consumers are generally justified by demonstrating that they are close in terms of equilibria, comparative statics, and welfare properties to models with a large but finite number of consumers, those models that one might believe are closer to reality. In this way, the mathematical simplicity of the continuum model prevails and one can speak of a pure or ideal form of perfect competition.

The focus of this paper is on a class of continuum models specialized to the particular needs of urban economics and regional science in general. The class also includes some models used for local public goods and differentiated products. Since the classics of Alonso (1964) and Beckmann (1969), urban economists and regional scientists have employed models with a continuum of agents distributed over a continuum of locations or land. Two goods are consumed by each consumer: land (a differentiated commodity) and a mobile composite consumption good that is taken to be numeraire. Each consumer is required to reside and consume goods in exactly one location (see Wheaton (1979)). These models have been shown to be tractable in that they circumvent mathematical complications identified by ten Raa and Berliant (1985). They have empirical uses and also applications to the local public goods literature (see Brueckner (1979)). However, Berliant (1985) shows that the standard justification for models with a continuum of consumers (given above) does not apply to this class of location models. In particular, a continuum of consumers each holding a positive area of land in a Euclidean space is impossible. Furthermore, any sequence of economies with a finite number of consumers tending to a limiting economy with a continuum of agents has the property that the land holdings and endowments of consumers must tend to zero on average. Consequently, the utility derived by a consumer from any positive area of land must be infinite in the continuum economy, as consumers own densities. This contrasts with the Hildenbrand (1974) story, which is used for models without location, where average endowments and consumption are positive, but the fraction of total commodities consumed by an agent tends to zero. Thus, these particular continuum location models have the property that they are not approximates, in the usual sense, to reasonable, large, finite economies. Under this approximation, the equilibria and comparative statics of the continuum economy can be vastly different from those of finite economies.

The purpose of this paper is to examine another possible approximation or interpretation of this class of continuum location models suggested in the urban economics literature and by Aumann (1964). A continuum of identical consumers (of measure 1, say) of the continuum economy is assumed to represent directly the behavior of one large consumer of an economy with a finite number of consumers. That is, each consumer of an economy with a finite population is represented by a continuum of infinitesimal particles. Thus, we examine whether location economies with a continuum of consumers, whose total mass represents a finite number of agents, and economies with a direct representation of finitely many agents are similar. Two notions of similarity are employed. The economies are strongly similar if demands are equal; they are weakly similar if equilibria are equal. Equality of demand requires that the demand mappings have equal values for all prices. Equality of equilibria requires that the mappings have equal values at equilibrium prices only. Note that whenever a continuum model is used for comparative statics, the user is on firm ground only if the model is strongly similar to a finite model, i.e. demands are the same at all prices. Of more importance, demands in the two models must vary in the same way (at least locally around equilibrium) with respect to parameter variations such as changes in endowments. Since the main result of this paper is negative, the emphasis will be on weak similarity in order to strengthen the conclusions.

The results detailed below show that it is possible to find equilibrium prices of a continuum model that do not clear the markets of analogous finite models, and there are equilibrium prices of a finite model that do not clear the markets of an analogous continuum model. Thus, the models are not weakly similar. The particular examples are simple (e.g., linear or Leontief preferences) and they seem robust. As a byproduct of this research, examples of continuum economies with identical individuals and well-behaved preferences but no equilibrium were found. Thus, the first order conditions for the continuum model can be different from those of an analogous finite model, and they can also be vacuous. In a companion paper, Berliant, Papageorgiou, and Wang (1986) have counterexamples to the first and second welfare theorems for a similar continuum model. Hence, the continuum model of location theory does not possess the properties of many neoclassical models. For a location model with a finite number of consumers that does possess these properties, see

Berliant (1985).

The papers of the urban economics literature most closely related to ours are Fujita and Smith (1986) and Scotchmer (1985). These papers demonstrate the existence of an equilibrium for location models with a continuum of consumers and a continuum and finite number of locations, respectively. Fujita and Smith consider an open model in which land is initially owned by nobody and rents exit the system. Furthermore, location cannot enter into a consumer's utility. In the urban literature, location usually enters into the utility function to account for the disutility of travel to work (see Wheaton (1977)). We shall return to this in the conclusion. Scotchmer considers a closed model in which location can enter into utility functions. The assumptions on utility functions include several beyond those normally used in general equilibrium theory, such as utility is zero if and only if land consumption is zero. These assumptions rule out many standard utilities, violate the conventional boundary condition on preferences, and do not reduce to standard assumptions when land is completely homogeneous. However, they also rule out the examples below. Thus, one might conjecture that these additional requirements are weakly necessary for the existence of equilibrium or perhaps an approximation theorem.

With regard to models of product differentiation, by interpreting the location attribute of urban economics as a general hedonic attribute or quality (see Lancaster (1966) or Rosen (1974)) one can use the continuum model detailed below as a model of product differentiation by reinterpreting the variables. This model is different from, say, those of Mas-Colell and Jones in the respect that consumers are restricted to the ownership of divisible goods in one location or of one quality. Many other location or product differentiation models focus on the firm or supply side of the market (see, for example, Fujita and Thisse (1986) or Novshek (1980)), which is passive in the model discussed below. However, if one allows consumers to be mobile with preferences over location or quality, then it is very likely that the examples below apply even with an active supply side of the market.

In general, we expect these examples to apply in any model where a continuum of consumers is mobile and restricted to choose one commodity out of a continuum. The intuition for this is not hard to understand. The equilibrium price or rent function on the differentiated commodity is expected to do too much in the location model. At each location, it must be equal to the marginal rate of substitution between land and numeraire. Furthermore, it must prevent the movement of consumers between locations. Generally, it cannot perform both functions simultaneously. In the finite model of Berliant (1985), the functions collapse into one.

The next section presents the models of the finite economies and of the continuum economies. The association of a particular finite model with a continuum model can be ambiguous. Section III presents a discussion of the various ways this can be done. Section IV contains the examples, while Section V concludes.

II. The Models

Land is a compact set in a Euclidean space. For the simplicity of the examples, it is taken to be a one-dimensional interval, a practice common in urban economics and regional science. Without loss of generality, the interval is $[-1,+1]$. If B is a Lebesgue-measurable subset of $[-1,1]$, let $|B|$ be its Lebesgue measure. When the urban economic concept of a Central Business District (CBD) is referenced, the definition shall be the mid-point, 0.

Following Wheaton (1979), the continuum economy has N types of consumers (N integer and finite) indexed by i, j, and k. Associated with type i is an interval $[0,1]$ of identical consumers of that type. The location of a consumer is indexed by $r \in [-1,1]$. For example 4 below we use $[0,1]$ to avoid the use of $|r|$ instead of r and to avoid factors of 2. Let $\mathbb{R}_+$ be the non-negative part of the real line. At a given location, a consumer has the non-negative orthant of $\mathbb{R}^2$ (call it $\mathbb{R}^2_+$) as the consumption set. The two goods are land, which for simplicity of the examples is assumed to be homogeneous across locations except for the locational attribute, and a composite consumption good. A consumer of type i has an endowment $y_i > 0$ of the consumption good but no land. Let the quantity of land consumption be given by h and the quantity of composite good consumption be given by x. For a given location r, preferences of a type i consumer are given by a quasi-concave, continuous utility function $u_i(r,h,x)$. Quasi-concavity in r is not assumed in order to be consistent with the location literature. Note that agents are endowed with (infinitesimal) fractions of income, just as they will consume fractions of land. If agents were endowed with positive amounts, the problems in Berliant (1985) appear.

We now may proceed to define an equilibrium concept. The composite consumption good will be used as numeraire since it is freely mobile and hence has the same price at all locations. To be determined in equilibrium are a rent density, $p(r) \in L^1([-1,1])$, giving the price of land at all locations; $h_i(r), x_i(r) \in L^1([-1,1])$, $i = 1,\dots,N$, giving the density consumptions of land and composite good, respectively, for each type at each location; and a density $m_i(r) \in L^1([-1,1])$ of consumers of type i residing at r. It is assumed that all consumers of type i at distance r from the city enter consume the same quantity of land, $h_i(r)$, and numeraire, $x_i(r)$. Alternatively, these quantities can be viewed as the <u>average</u> holdings of such consumers. This is consistent with the urban economics literature, e.g. Wheaton (1979). Consumers of type i face the following maximization problem:

$$\underset{r,h,x}{\text{Maximize}} \quad u_i(r,h,x) \tag{1}$$

$$\text{subject to} \quad p(r)h + x \leq y_i \tag{2}$$

The solutions to this problem are called <u>continuum economy demand</u>. Demand can be empty if prices are discontinuous, but the examples below avoid this problem and display nonempty demand at equilibrium prices.

We do not define a feasible allocation for the reason that the models in the literature tend to be open rather than closed. No agent explicitly owns the land initially, while the rents paid for land exit the system. This is a valid simplification, since it is easy to add landlords, one at each point, who own all land initially and derive utility only from composite good. In fact, this closure of the model is used by Fujita (1986) and Berliant, Papageorgiou, and Wang (1986) to examine the welfare theorems.

A <u>continuum</u> <u>economy</u> <u>equilibrium</u> with respect to endowments $\{y_i\}_{i=1}^N$ is a collection of densities p^*, $(m_i^*, h_i^*, x_i^*)_{i=1}^N \in L^1$ such that:

- For almost every $r \in [-1,1]$, for each i with $m_i(r) > 0$, $r, h_i^*(r), x_i^*(r)$ maximizes (1) subject to (2) at prices p^*.
- $\int_{-1}^{1} m_i^*(r) = 1$ for all i.
- For almost every $r \in [-1,1]$, $\sum_{i=1}^{N} m_i^*(r) h_i^*(r) \leq 1$.

The first condition states that consumers are maximizing subject to their budgets. The second condition states that all consumers are located. The final condition assumes that the land available at each location is one unit, for simplicity; this can easily be modified to accomodate $2\pi r$ as in the plane-location literature. Since $m_i^*(r)$ is population density and $h_i^*(r)$ is mean land density with respect to location, their product is equal to the land density demand of type i consumers, so the last equilibrium condition is the land market clearance requirement. For composite good endowments $(y_i)_{i=1}^N$, the set of equilibria is denoted $e(y_1, \ldots, y_N)$.

Several aspects of this continuum model deserve discussion. First, for most models with prices in L^1, prices are defined only almost surely. Changing the prices at one point will lead to an "equivalent" set of prices that clear markets (almost surely). For example, the model with a finite number of consumers detailed below has this property. The continuum model above does not have this property. If the equilibrium price of land at one location is changed to zero, all consumers will reside there (provided preferences are monotonic in land) and there will no longer be equilibrium. Thus, there seem to be some deep measure-theoretic problems with the model.

Second, one can have $h_i(r) > 1$ for some i and r in equilibrium, provided $m_i(r) < 1$. The interpretation of this inequality is unclear, as it seems to suggest that average or per capita land holdings can exceed supply at r, even on a set of positive measure.

Third, it is standard to put transportation costs into this model, e.g. change the budget constraint to $x + p(r)h + |r| \leq y_i$. For simplicity, our examples below do not involve such costs, although they could be included in a neutral way by the following trick. In the model, (1,2), take a new utility function, say v_i, defined by $v_i(r,h,x) = u_i(r,h,x - |r|)$.

The problem

maximize $\quad v_i(r,h,x)$

subject to $\quad p(r)h + x \leq y_i$

reduces to:

maximize $\quad u_i(r,h,x')$

subject to $\quad p(r)h + x' + |r| \leq y_i$

where $x' = x - |r|$. Hence model (1,2), though simplified, covers the standard model of the new urban economics.

Finally, the continuum model presented above suppresses two externalities that appear in the standard models of urban economics and regional science. The consumers in the models of this literature take the locations of each other into account. At each location, the population density is inverted in order to calculate land availability (supply). This is the amount of land that consumers must consume at any location. Thus, supply enters into the decision problem of the consumers. This externality is essentially a reduced form feature of the equilibrium equations, and we prefer to present the model in structural form. The second externality is more explicit in that consumers may care about the population surrounding them (see Beckmann (1977) or ten Raa (1984)). In this case, the global population distribution is assessed in terms of proximity when social interaction is a good. The inclusion of such an externality would complicate the model and examples in an obvious way without adding further insights to the relationship with the finite model.

Turning now to the <u>finite economy</u>, there are N consumers, one for each type of the continuum model. Consumer i is endowed with income or composite good $y_i \in \mathbb{R}_+$, the total of the incomes of those consumers it represents. The consumption set of each consumer is $\mathcal{B} \times \mathbb{R}_+$, where $\mathcal{B}$ represents the σ-algebra of measurable subsets of $[-1,1]$. Consumers of the finite economy actually buy subsets of land rather than densities, and can consume composite good as well. Preferences are represented by utility functions. The utility function of consumer i is a map $U_i: \mathcal{B} \times \mathbb{R}_+ \to \mathbb{R}$. We write $U_i(B,x)$ for a parcel $B \in \mathcal{B}$ and (residual) composite good $x \in \mathbb{R}_+$. Prices are, once again, densities in L^1. Consumers pay the integral of the price density over any parcel (see Berliant and ten Raa (1986)). For given $p \in L^1$, the problem of consumer i is:

$$\underset{B \in \mathcal{B},\ x \geq 0}{\text{Maximize}} \quad U_i(B,x) \tag{3}$$

$$\text{subject to} \quad \int_B p(r)dr + x \leq y_i \tag{4}$$

The solutions to this problem are called <u>finite economy demand</u>. As for continuum economy demand, this demand can also be empty. If utility or preferences are continuous with respect to the topology specified in Berliant and ten Raa (1986), demand is nonempty.

A <u>finite economy equilibrium</u> is a price system $p \in L^1$ and $(B_i, x_i)_{i=1}^N \in (\mathscr{B} \times \mathbb{R}_+)^N$ such that (B_i, x_i) maximizes (3) subject to (4) for each i and $(B_1, B_2, \ldots, B_N)$ partitions [-1,1]. For composite good endowments $(y_1, \ldots, y_N)$, the set of equilibria is denoted $E(y_1, \ldots, y_N)$. More details about this model can be found in Berliant (1985).

III. The Association of Models

The data of an economy, be it modelled in the finite or in the continuum mode, consist of endowments and utility functions. A finite and a continuum model represent a common economy if endowments are equal and utility functions equivalent. We cannot require equality of the respective utility functions, simply because the domains differ. In the finite model, utility is defined on $\mathscr{B} \times \mathbb{R}_+$; in the continuum model, utility is defined on $([-1,1] \times \mathbb{R}_+) \times \mathbb{R}_+$. Land enters utility through a parcel, $B \in \mathscr{B}$, in the finite case, and through a location-quantity pair, $(r,h) \in [-1,1] \times \mathbb{R}_+$, in the continuum case. The latter approach is more restrictive, since location-quantity features are also captured by commodities $B \in \mathscr{B}$, but these parcels embody other qualities as well (such as shape). Thus, in studying equivalent utility functions, it is natural to start with a continuum economy utility function, u, and to associate a finite economy utility function, U. Since U is defined on a much larger commodity space, the extension will not be unique.

Fix $u:([-1,1] \times \mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$. Take a commodity of the finite model, $(B,x) \in \mathcal{B} \times \mathbb{R}_+$. The question is how much utility must be associated with it. In the continuum model, a consumer is represented by a mass distribution, each infinitesimal particle of which is an agent who consumes densities of land and numeraire commodity to collect a density of utility value. Now the very concept of a density suggests additivity. Therefore, it is natural to distribute the mass of the consumer, 1, and the numeraire, x, across the parcel under consideration, B; evaluate utility densities and integrate them. Let the consumer and numeraire distributions be m and z, respectively. Assume m(r) and z(r) are positive only if $r \in B$; furthermore, $\int_B m(r)dr = 1$ and $\int_B z(r)dr = x$. Note that mean numeraire density at r is $z(r)/m(r)$, mean land density is $1/m(r)$, mean utility density is $u[r,1/m(r),z(r)/m(r)]$, and utility density is $u[r,1/m(r),z(r)/m(r)]m(r)$. Hence, parcel B carries total utility $\int_B u[r,1/m(r),z(r)/m(r)]m(r)dr$. It remains to specify m(r) and z(r); they fix total utility, U(B,x). This problem is analogous to the construction of a social welfare function. We consider the traditional constructs, the utilitarian and the egalitarian functions. We also consider a construct suggested to us by David Pines, namely the market function, in which individuals are forced to decentralize their pieces and consume Walrasian allocations. Since we construct the amount of utility an individual can get out of a commodity by integration of utility densities processed by fractions of the individual, the utilitarian construct seems to be the appropriate one.

Utilitarian utility is defined by

$$U(B,x) = \sup_{m,z} \int_B u[r,1/m(r),z(r)/m(r)]m(r)dr$$

subject to

$$\int_B m(r)dr = 1 \text{ and } \int_B z(r)dr = x.$$

Egalitarian utility is defined as the solution to the same problem but with the additional constraint that mean utility density is constant: $u[r,1/m(r),z(r)/m(r)]$ = constant wherever $m(r) > 0$. Note that the value of the egalitarian utility will be equal to the constant, since the total consumer mass is one. David Pines' market utility is defined to be the equilibrium utility level that is obtained by the continuum economy with a single type of consumer, $N = 1$, and land endowment B instead of $[-1,1]$.

Note also that when the utility density function is linear homogeneous at each location, the mass density cancels and total utility reduces to

$$U(B,x) = \sup_{z} \int_B u[r,1,z(r)]dr$$

subject to

$$\int_B z(r) = x$$

for the utilitarian case. In this case, the maximization problem fixes the spatial income density z, whereas the mass density m can be chosen freely. In particular, it can be chosen to equalize mean utility across locations. This is an example in which the utilitarian and egalitarian utilities coincide.

Since this coincidence will also hold for the examples in the next section, the ambiguity of the association of continuum and finite economy models is restricted. By separate consideration of the remaining utility construct, Pines' market function, this ambiguity is not used as a source for examples of economies that are not weakly similar.

Now that we have specified the association of finite to continuum models that will be used, other possible associations can be compared to it. First, some general remarks are in order. It is only necessary to associate a finite model with a given continuum model, and not vice-versa, since we only seek to try to justify a continuum model by generating an appropriate finite model. The inverse mapping is not relevant to this problem.

There are many other ways to associate a finite model with a continuum model. What led us to this particular association was a number of factors. First, the utilitarian social welfare function is used frequently in the urban economics literature (see, for example, Mirrlees (1972)). Second, a discussion with Aumann suggested that he intended in his early work that consumers be of arbitrarily small but positive measure. A natural way to take such limits is through a utilitarian welfare function. Finally, we have also considered many other alternatives, and found none as appealing as the association given above.

For example, one can use a "locator function" approach to association. To be precise, let continuum model utility be given by $u(r,h,x)$. If $B \in \mathscr{B}$, we may define $U(B,x) = u(\ell(B),|B|,x)$, where $\ell\colon \mathscr{B} \to [-1,1]$. The map ℓ locates a consumer of the finite model in his plot of land, and is called a locator function.

This approach is quite general, and does not specify a particular locator function corresponding to any given utility function for the continuum model. In fact, it does not seem possible a priori to distinguish various locator functions from one another. Nonetheless, we have found that the examples generated in the next section can usually be interpreted or modified so as to apply to other techniques for associating models, such as the locator function approach. We will amplify this point for example 1 below.

Clearly, there is a large number of ways to associate finite models with continuum models. Of course, it is not possible to show, using examples, that there is no way to associate them so that they are similar. On the other hand, it is not obvious that proving a general theorem is worth the trouble and complexity that it would involve. This belief stems from our experience that the locator function inducing similarity depends on the utility function. So, once the locator function is fixed, alteration of the utility function will produce examples of dissimilarity.

IV. Examples

In this section, we present a number of continuum economies, specified by endowments and utility functions, and their associated finite economies. In the first two examples, the continuum economy equilibrium exists and is unique, but the finite economy has many equilibria (example 1) or none at all (example 2). In example 3, the continuum economy has no equilibrium, but the finite economy has one. It is artificial: generally existence of equilibrium in an associated finite economy implies existence of equilibrium in the underlying continuum economy. This relationship is not useful, though, in view of our final result. Example 4 shows that even when equilibria exist for both models, they can be different.

Example 1. For the first example, two types of agents have utility functions $u_1(r,h,x) = h + x$ and $u_2(r,h,x) = |r|h + x$, respectively.

Let us first solve for the continuum model equilibrium. Type 1 agents maximize $h + x$ subject to $p(r)h + x \leq y_1$. Being indifferent between locations, they consider only the support of the minimum of $p(r)$. If the minimum price is less than one, $p^{min} < 1$, then $x = 0$ and $\int h(r)dr = \frac{y_1}{p^{min}}$ on the support of the minimum. If $p^{min} > 1$, then $h = 0$ and $x = y_1$. In the hairline case $p^{min} = 1$, demand is multivalued as agents are indifferent between land and numeraire commodity. To type 2 agents, the marginal utility of land is lower, especially towards the CBD. Their marginal cost benefit ratio is $\frac{p(r)}{|r|}$. If the minimum value of this ratio is less than one, then $x = 0$ and $\int h(r)dr = \frac{y_2}{p(r)}$ on the support of the minimum. If the minimum value exceeds one, then $h = 0$ and $x = y_2$. This completes the derivation of demand. Note that wherever $p(r) > 1$, demand for land is zero, which falls short of supply. Hence, in equilibrium, $p \leq 1$. Consider a location, r, where $p(r) = p^{min}$, and all points closer to the center: $|s| < |r|$. If $p(s) > p(r)$, then no type 1 agent will demand land at s and $\frac{p(s)}{|s|} > \frac{p(r)}{|r|}$, so that no type 2 agent will demand land at s either. Hence, $p(s) \leq p(r) = p^{min}$, and therefore $p(s) = p^{min}$. It follows that in equilibrium, p is minimized precisely on an interval, $[-r*,r*] \subseteq [-1,1]$. Now consider the other points s, $|s| \geq r*$. Since no type 1 agent will demand land at such points in equilibrium, this land must be demanded by type 2 agents. Hence $\frac{p(s)}{|s|}$ must be minimal at any s of $[-1,-r*] \cup [r*,1]$, and $p(s) \leq 1$. Hence, $p(s) = \frac{p^{min}}{r*}|s|$, $|s| > r*$. The coefficient $\frac{p^{min}}{r*}$ is pegged by the continuity of p. (If p were discontinuous, then a jump in utility levels across locations would be possible, which is not true in equilibrium.) Note that $r* = 0$ would attract all type 1 agents and yield excess demand at 0. Hence, equilibrium prices are dish-shaped.

It remains to fix r* and p^{min}. Note that type 1 agents will demand $r \in [-r*,r*]$ and type 2 agents will demand $s \in [-1,-r*] \cup [r*,1]$. Also, $p^{min} \leq r*$, for if $1 \geq p^{min} > r*$, then there is excess supply of land for $|s|$ sufficiently close to 1. If $p^{min} < r* = 1$, there is excess demand for land at r*. Thus, we have $p^{min} \leq r* \leq 1$ but not $p^{min} < r* = 1$. Distinguish the three remaining cases:

A) $p^{min} = 1$

B) $p^{min} < r* < 1$

C) $p^{min} = r* < 1$.

A. $p^{min} = 1$. Since in equilibrium $p \leq 1$, it must be that $p = 1$. Hence, $r* = 1$. Type 1 agents are indifferent between any h(r) and x. Type 2 agents are indifferent between h(-r*), h(r*) and x. In equilibrium, almost all land is bought by type 1 agents. Hence, residual income amounts to $x_1 = y_1 - 2$, where the latter component is the value of all land. Note that this case is relevant only if $y_1 \geq 2$.

B. $p^{min} < r* < 1$. In the same manner as for case A, $x_1 = y_1 - 2p^{min}r*$ and $x_2 = y_2 - 2\int_{r*}^{1} \frac{p^{min}}{r*} s\, ds = y_2 - \frac{p^{min}}{r*}(1-r*^2)$. In this case, however, the marginal cost benefit ratio is less than one, so that agents want to spend all money on land: $x_1 = x_2 = 0$. It follows that $2p^{min}r* = y_1$ and $\frac{p^{min}}{r*}(1-r*^2) = y_2$. Consequently, $r* = (\frac{y_1}{y_1+2y_2})^{1/2}$ and $p^{min} = (1/2)\sqrt{y_1(y_1+2y_2)}$. Note that this case is relevant only if $(1/2)\sqrt{y_1(y_1+2y_2)} < (\frac{y_1}{y_1+2y_2})^{1/2} < 1$; that is, using positivity of the endowments, $y_1 + 2y_2 < 2$.

C. $p^{min} = r* < 1$. Type 1 agents want to spend all their money on land: $2p^{min}r* = y_1$. Type 2 agents are indifferent between land in $[-1,-r*] \cup [r*,1]$ and composite good. $\frac{p^{min}}{r*}(1-r*^2) \leq y_2$. It follows that $p^{min} = r* = (\frac{y_1}{2})^{1/2}$ and $1 - \frac{y_1}{2} \leq y_2$ and $y_1 < 2$.

Taken together, the equilibrium price is as follows. If $y_1 < 2 - 2y_2$, then p is dish shaped with $r* = (\frac{y_1}{y_1+2y_2})^{1/2}$ and $p^{min} = (1/2)\sqrt{y_1(y_1+2y_2)}$ (which is less than r*). If $2 - 2y_2 \leq y_1 < 2$, then p is dish shaped with $r* = p^{min} = (\frac{y_1}{2})^{1/2}$. If $y_1 \geq 2$, then $p = 1$.

Now let us turn to the associated finite economy. Since utility densities are linear homogenous at each location, the simplified expression for total utility, be it utilitarian or egalitarian, from the last section is:

$$
\begin{aligned}
U_1(B,x) &= \sup_z \int_B u[r,1,z(r)]dr \\
&= \sup_z \int_B [1+z(r)]dr \\
&= |B| + \int_B z(r)dr \\
&= |B| + x
\end{aligned}
$$

and

$$
\begin{aligned}
U_2(B,x) &= \sup_z \int_B [\,|r| + z(r)]dr \\
&= \int_B |r|dr + x.
\end{aligned}
$$

It is not difficult to verify that the continuum equilibrium price does clear the finite economy. This is essentially due to the absence of income effects at every location. Therefore, perfect aggregation conditions are fulfilled and the aggregate consumers are representative of the infinitesimal agents of the continuum economy. However, there are many more equilibria for the finite economy. Any rent gradient increasing as one moves away from the CBD will do. For example, $p(r) = |r|$ clears the finite economy. The first consumer will demand $[-r*, r*]$ since any point yields more benefit than cost, where $r*$ is as big as the budget allows, i.e.

$\int_{r*}^{r*} |r|dr = y_1$, or $r* = \sqrt{y_1}$ if $y_1 \leq 1$ and $r* = 1$ otherwise. Consumer 2 is completely indifferent between owning land at any location and numeraire. Hence, he can choose the remainder of land.

Under the locator function approach, the finite economy utilities are

$|B| + x$ and

$|\ell(B)||B| + x$, respectively.

Assuming that $\ell(B)$ is continuous with respect to the topology of Berliant and ten Raa (1986), demand is nonempty for both consumers. If $y_1 < 2$ but close to 2, then if an equilibrium price exists for the finite economy, it is not dish-shaped as described above (case C). For if it were dish-shaped, then consumers of type 2, who occupy the outer edges of the dish, can increase their utility by buying a little more land. All that is needed for this is that ℓ is Lipschitz in a weak sense. Then the gain in $|B|$ has a benefit intensity of more than r^*, which equals the marginal cost in this case (a detailed proof is available from the authors).

Example 2. Now we will present a continuum economy with an equilibrium whose price does not clear the finite economy. Consider

$$u(r,h,x) = |r| + \sqrt{hx}.$$

To solve for equilibrium in the continuum economy, consider the consumer's problem of maximization of $|r| + \sqrt{hx}$ subject to $p(r)h + x \leq y$. At each location r, first order conditions and the budget yield the optimum quantity of land to purchase: $h(r) = \frac{y}{2p(r)}$, while residual income is spent on numeraire. Substituting back into utility we obtain the utility level $|r| + (\frac{y}{2p(r)}\frac{y}{2})^{1/2} = |r| + \frac{y}{2\sqrt{p(r)}}$. In equilibrium, this utility level must be constant across locations: $|r| + \frac{y}{2\sqrt{p(r)}} = u_o$. Consequently, the equilibrium price is $p(r) = \frac{y^2}{4(u_o - |r|)^2}$. (The constant u_0 is determined by the land clearance condition, $\int \frac{1}{h(r)} dr = N$, where N is the number of consumers. In fact, $u_0 = 1/2 + (\frac{y}{N} + \frac{1}{4})^{1/2}$.)

Turn to the associated finite economy. By the formula of section III (utilitarian) utility amounts to

$$U(B,x) = \sup_{m,z} \int_B u[r, 1/m(r), z(r)/m(r)]m(r)dr$$

subject to

$$\int_B m(r)dr = 1 \text{ and } \int_B z(r)dr = x.$$

Hence,

$$\begin{aligned} U(B,x) &= \sup_{m,z} \int_B [\,|r| + (\frac{1}{m(r)} \frac{z(r)}{m(r)})^{1/2}]m(r)dr \\ &= \sup_{m,z} [\int_B |r|m(r)dr + \int_B \sqrt{z(r)}dr] \\ &= \sup_{m} \int_B |r|m(r)dr + \sup_{z} \int_B \sqrt{z(r)}dr \\ &= \|B\| + (\frac{x}{|B|})^{1/2}|B| = \|B\| + \sqrt{|B|x} \end{aligned}$$

where $\|B\| = \sup\{|r| \mid r \in B\}$ and, as before, $|B|$ is the measure of B. The limits of m and z that yield the supremum value for the last two terms are the density of the distribution concentrated on sup B or inf B and the constant, $\frac{x}{|B|}$, respectively. Since all mass of the consumer distribution is concentrated on a single point, this utility is also egalitarian. Clearly, wherever price is locally integrable, all consumers will demand the boundary points ± 1. This is particularly true for the continuum equilibrium price. Thus, no equilibrium exists for the finite economy. The reason is that this utility is not continuous in the sense of Berliant and ten Raa (1986).

<u>Example 3:</u>

The next example is one for which no continuum model equilibrium exists, but when the associated finite model is formulated, an equilibrium does exist for it. This is further evidence that the continuum model cannot be approximated, although we admit that the example is artificial in the choice of utility functions. We shall discuss the implications after presentation of the example. Let

$$u(r,h,x) = h - |r| \qquad \text{if } h \leq |r|$$

and $$u(r,h,x) = (1 - |r|/h)x \qquad \text{otherwise.}$$

u is continuous in all of its arguments and concave for given r. It also exhibits local nonsatiation, but is not monotonic. Monotonicity is inessential in standard general equilibrium analysis, but its absence does harm existence of equilibrium for the continuum model.

Select any potential equilibrium price system, $p(r)$. For each r, distinguish two cases:

A. $p(r) \geq y/|r|$

B. $p(r) < y/|r|$

In case A, since the budget constraint is $p(r)h + x \leq y$, if $h > |r|$ then $p(r)h + x > p(r)|r| + x \geq y$ ($p(r) = 0$ is clearly not part of an equilibrium), which is a contradiction. So for case A, it must be that $h \leq |r|$. In this case, the utility function forces agents to spend all on land (h), so that $h = \frac{y}{p(r)}$ and $x = 0$. In case B, $|r|p(r) < y$, so if $h \le r$, then $hp(r) < y$. Hence some income is spent on numeraire, $x > 0$. This is not rational if $h \le |r|$, so it must be that $h > |r|$. In this case, there is an interior solution obtained from the first order conditions and the budget constraint:

$$h = [|r|y/p(r)]^{1/2}, \quad x = y - [|r|p(r)y]^{1/2}.$$

Note that demand is continuous even at $p(r) = y/|r|$. An equilibrium condition is that the utility level must be the same, say u_o, across inhabited locations, for otherwise consumers would move. Consider first case A at some location r. Then $h \le |r|$ and $u_o = \frac{y}{p(r)} - |r|$, or $\frac{y}{u_o + |r|} = p(r) \ge \frac{y}{|r|}$. Hence $u_o \le 0$. If case B holds at r, then $h > |r|$ and

$$u_o = \{1 - [|r|p(r)/y]^{1/2}\}\{y - [|r|p(r)y]^{1/2}\} = \{1 - [|r|p(r)/y]^{1/2}\}^2 y > 0,$$

since $|r|p(r) < y$ in this case. Since u_o is a constant, it must be that either case A holds everywhere or case B holds everywhere. If case A holds in every inhabited location, then consumers can always move to r=0 and obtain strictly positive utility, so the original allocation was not an equilibrium. If case B holds in every inhabited location,

$$u_o = \{1 - [|r|p(r)/y]^{1/2}\}^2 y.$$

Solving,

$$p(r) = (y^{1/2} - u_o^{1/2})^2/|r|$$

$$h(r) = [|r|y/p(r)]^{1/2} = |r|y^{1/2}/(y^{1/2} - u_o^{1/2})$$

Given that $m(r)h(r) = 1$ a.s., it must be that

$$\int_{-1}^{1} \frac{1}{h(r)} dr = \int_{-1}^{1} m(r)dr = N. \quad \text{But}$$

$$\int_{-1}^{1} \frac{1}{h(r)} dr = \int_{-1}^{1} (y^{1/2} - u_o^{1/2})/(y^{1/2}|r|)dr.$$

The last expression is zero if $y = u_o$ and undefined otherwise as $1/|r|$ is not integrable at 0. In either case, this expression is not equal to $N > 0$. Thus, there is no equilibrium for the continuum economy.

Turn to the associated finite economy.

$$U(B,x) = \sup_{m,z} (\int_{A^*} + \int_{B^*})u[r,1/m(r),z(r)/m(r)]m(r)dr$$

subject to $\int_B m(r) = 1$ and $\int_B z(r)dr = x$.

Here $A^* = \{r \in B \mid m(r) \geq 1/|r|\}$

and $B^* = \{r \in B \mid m(r) < 1|r|\}$

Substituting the expression for u,

$$U(B,x) = \sup_{m,z} \{\int_{A^*}[\frac{1}{m(r)} - |r|]m(r)dr + \int_{B^*}[1-|r|m(r)]z(r)dr\}$$

The first integrand is nonpositive. Hence

$$U(B,x) \leq \int_{B^*}[1 - |r|m(r)]z(r)dr \leq \int_{B^*}z(r)dr \leq \int_B z(r)dr = x.$$

It is not necessary to write down U(B,x) precisely. We see straightaway that, for example, $p = 0$ is an equilibrium. Consumers' demand includes (ϕ,y) since its utility equals the previously established upper bound. Supply of land also includes the empty set when rent is zero. A few remarks are in order. Egalitarian utility does not always exist in this example. Its existence requires that B is contained in either A^* or B^*, so it is not defined for all B. Pines' utility does not exist, for it is based on continuum equilibria.

These are indications of the difficulty of associating a finite economy that admits an equilibrium with a continuum economy that does not. In fact, if continuum utility is assumed continuous and monotone, then either the associated finite economy utility function is discontinuous in the Berliant-ten Raa (1985) topology or always infinite, or continuum equilibrium can be demonstrated to exist by Lebesgue's Monotone Convergence Theorem. So continuity and monotonicity render the construction of an example without a continuum equilibrium but with a finite equilibrium hopeless. The logical negation yields a positive statement: If a finite economy with a continuous and monotone (density-based) utility function has an equilibrium, then so does the underlying continuum economy. This suggests that some continuum economy equilibria might be approximated by finite equilibria. The next example shows, however, that when equilibria exist in both models, they can be very different.

Example 4. So far we have seen that continuum and associated finite models can be incompatible by mere lack of existence of equilibrium in one of the two. The next example shows that the existence issue is not critical. We will now present an example with different equilibrium price systems for the continuum and the finite models. Consider $r \in [0,1]$ and $u(r,h,x) = hx^r$.

First we determine the equilibrium price for the continuum economy. Using the first order conditions, at location r, the optimum quantity of land to purchase is $h(r) = \frac{y}{(1+r)p(r)}$, while residual income is spent on the numeraire good. Substituting back we obtain the utility level $\frac{y}{(1+r)p(r)}(y - \frac{y}{1+r})^r = \frac{r^r}{(1+r)^{1+r}} \frac{y^{1+r}}{p(r)}$. In equilibrium this must be equal to a constant, say u_o, across locations. Hence, the equilibrium price is

$$p(r) = \frac{r^r}{(1+r)^{1+r}} \frac{y^{1+r}}{u_o} \quad \text{[N.B. } r \downarrow 0 \text{ implies } p(r) \to \frac{y}{u_o}\text{]}.$$

Here the constant u_o is determined by the land market clearance condition $\int_0^1 \frac{1}{h(r)}\,dr = N$ or, substituting the expressions for h(r) and p(r),

$$u_o = \frac{1}{N}\int_0^1 \left(\frac{ry}{1+r}\right)^r dr.$$

Turn to the associated finite economy. By the formula of section III, (utilitarian) utility amounts to

$$U(B,x) = \sup_{m,z} \int_B u[r,1/m(r),z(r)/m(r)]m(r)dr$$

subject to

$$\int_B m(r)dr = 1 \text{ and } \int_B z(r)dr = x.$$

Hence,

$$U(B,x) = \sup_{m,z} \int_B \left[\frac{z(r)}{m(r)}\right]^r dr$$

subject to

$$\int_B m(r)dr = 1 \text{ and } \int_B z(r)dr = x.$$

Let λ and μ be the Lagrange multipliers associated with the first and second constraints, respectively. Then the first order conditions are

$$-rm(r)^{-r-1}z(r)^r = \lambda$$

$$rz(r)^{r-1}m(r)^{-r} = \mu$$

Solving, we obtain

$$m(r) = (-\lambda)^{r-1}\mu^{-r}r$$
$$z(r) = (-\lambda)^{r}\mu^{-1-r}r$$

Note that $z(r) = \frac{\lambda}{\mu}m(r)$. Integration and application of the constraints yields $x = \frac{\lambda}{\mu}$.

It follows that

$$U(B,x) = \int_B [\frac{z(r)}{m(r)}]^r dr = \int_B (\frac{\lambda}{\mu})^r dr = \int_B x^r dr.$$

Note that mean utility density is

$$u[r,1/m(r),z(r)/m(r)] = u[r,1/m(r),x] = \frac{x^r}{m(r)} = (-\lambda)^{1-r}\mu^r x^r/r.$$

For $x > 0$ this varies with r. Hence, the utility function of the finite economy seems nonegalitarian. Yet the utility function is egalitarian. To prove this claim we must shuffle agents and numeraire commodity without reducing utility. This is possible because of a multicollinearity between the two. The crux of the first order conditions is $z(r) = xm(r)$. Whenever this holds, $\int_B u[r,1/m(r),z(r)/m(r)]m(r)dr = \int_B x^r dr = U(B,x)$. Mean utility density is $\frac{x^r}{m(r)}$. So, simply put, $m(r) = cx^r$ with c determined by the constraint $\int_B m(r)dr = 1$, i.e. $c = \frac{1}{\int_B x^r dr}$. Then mean utility density is $\frac{x^r}{m(r)} = \frac{1}{c} = \int_B x^r dr$

and total utility equals utilitarian total utility, $\int_B x^r dr$. Note also that $U([r,r+h),x)$, $h \downarrow 0$, reduces to $x^r h = u(r,h,x)$, as should be.

To derive demand in the finite economy, it is convenient to fix x temporarily and to determine the best B(x) that goes with it. This is a straightforward application of Berliant (1984). Then we must maximize $U[B(x),x] = \int_{B(x)} x^r dr$. Since this is continuous from above, it is maximized by some x*. Thus, [B(x*),x*] is demanded. Its existence can also be established by Berliant and ten Raa (1986). Now let N > 1. Then, to avoid excess demand, the Berliant (1984) benefit cost ratio must be constant at x*. Consequently, $p(r) = c^* x^{*r}$. Here c* is determined by Walras law, $c^* \int_0^1 x^{*r} dr + Nx^* = Ny$. Hence, $c^* = N\frac{(y - x^*)}{\int_0^1 x^{*r} dr} = N\frac{y - x^*}{x^* - 1} \log x^*$.

The continuum and finite economy prices are $p(r) = \frac{r^r}{(1+r)^{1+r}} \frac{y^{1+r}}{u_o}$ and $p(r) = c^* x^{*r}$, respectively. Suppose they are equal. Then (putting r = 0) $\frac{y}{u_o} = c^*$, hence, $\frac{r^r}{(1+r)^{1+r}} c^* y^r = c^* x^{*r}$, or $\frac{r}{(1+r)^{1+(1/r)}} y = x^*$, or $\frac{r}{(1+r)^{1+(1/r)}} = \frac{x^*}{y}$, which cannot hold for all r. Hence, the equilibrium price systems differ.

David Pines' market utility function is easily found by recalling the expression for u_o in the continuum model. Thus, we define the (finite model) utility of a parcel and numeraire to be the equilibrium utility attained by a continuum of consumers (of measure 1) with that numeraire locating on that parcel:

$$u(B,x) = \int_B \left(\frac{rx}{1+r}\right)^r dr.$$

The consequent finite model equilibrium price is

$$p(r) = c^{**}\left(\frac{rx^{**}}{1+r}\right)^r$$

where the constants are determined precisely as in the utilitarian case. Interestingly enough, David Pines' equilibrium price is different from both the continuum model equilibrium price and the utilitarian/egalitarian finite model equilibrium price.

Four remarks round off this example. First, if the number of consumers increases, all equilibrium prices increase proportionally, so their relative differences persist. Second, the income effects in this example do not form the wedge between the continuum and the finite model. The difference persists if we use the example $u(r,h,x) = h^{1-r}x^r$. Third, the crux of the difference between the continuum and finite models is that, for the continuum model, the marginal utility of numeraire is allowed to vary freely in one location (with land consumption) without taking into account land consumption at neighbouring points. In the finite model, the marginal utility of numeraire is pegged by a land density of one and can vary only through cross elasticities with locations elsewhere. The consequent wedge between the marginal rates of substitution in the two models yields an equilibrium price difference that persists when the number of consumers increases. Lastly, since the utility function in the finite economy is essentially a welfare function for the consumers of the continuum economy, and since the associated finite economy price is a decentralization of the welfare optimum that apparently differs from the continuum equilibrium price, the welfare theorems may fail in the continuum economy. For detailed analysis, we refer to Berliant, Papageorgiou, and Wang (1986).

Further Examples: Returning now to the question of existence of an equilibrium of the continuum model, it was remarked above that the existence of equilibrium for this model is quite tight. This is true even when all of the standard assumptions used to prove that an equilibrium (or quasi-equilibrium) exists hold. For the sake of brevity, we shall not go through the details of the examples below (which are available from the authors), but simply list utility functions for which continuum economies with identical consumers do not have equilibria for a large set of endowments.

$$u(r,h,x) = \min(|r|,h,x)$$

$$u(r,h,x) = \min(r^2 x,h) \quad (r \in [0,1])$$

$$u(r,h,x) = rx + h \quad (r \in [0,1]) \quad Ny > \pi.$$

If one prefers decreasing utility as distance from the city center increases, minor alterations of any of the examples in this paper will suffice. The procedure has been detailed in Section II.

V. Conclusion

In a series of papers, we have seen that the standard continuum model of spatial economics is flawed because it cannot generally be approximated by finite models, and because equilibrium may fail to exist or the welfare theorems may fail even under standard assumptions of general equilibrium analysis. What distinguishes this model from others is that there is a continuum of consumers, each of whom must choose one of a continuum of locations (or qualities) at which to consume completely divisible commodities.

The major difference between the continuum and finite models is that the marginal utility for numeraire depends only on land purchases at a given location for the continuum model, while it depends on land purchases, of fixed quantity, at many locations for the finite model. Thus, marginal rates of substitution for the two models are not necessarily related.

The major difference between this type of continuum model and that of Mas-Colell (1975) or Jones (1983) is that the consumption set is not convex here due to the restriction to the choice of exactly one location, while commodities are divisible.

There are several other ways of putting this. First, as noted in the introduction, utility is associated with location as well as the other goods. Hence, equilibrium prices must be equal to the marginal rate of substitution at each location as well as prevent consumers from moving between locations. In this sense, there are not enough prices or incomplete markets, since location itself is not priced. Moreover, there is no requirement that utility be quasi-concave in location, if this is considered to be a good, and so no price support can be expected. If location simply indexes goods (in this case, h), then it is not obvious that utility can be extended to a larger, linear space (such as the space of distributions) in a quasi-concave manner. Furthermore, the equilibria of such an extended model will generally involve consumer purchases at many locations.

One point that this discussion brings out is the difference between location as a commodity in itself and location as an attribute of a commodity. For the differentiated products literature, one generally leans toward the latter interpretation, but this is not entirely obvious if consumers purchase goods at only one location. Does the effect of location stay fixed or change as more of the differentiated commodity is purchased? With regard to the spatial or location theory literature, one leans toward the former interpretation. This is clear from the additively separable form of utility in location and land (see Beckmann (1969)). Presumably, this comes from the assumption that weather, geography, and the utility cost of travel to work have a fixed effect on utility, at least to some degree. Is location a good in such models, and should it be priced?.

We wish to make four final points before summarizing the main message of this paper. First, the problems presented here arise primarily because location enters explicitly into consumers' utility functions. Second, it is easy to close both the finite and continuum models while retaining the results by using landlords who own all land initially but desire only composite consumption good. Third, it does not matter whether the point-by-point or aggregate market clearing condition is used for land, as the examples can be modified to accept either. Fourth, the examples of continuum models without equilibrium extend easily to a finite or countable number of locations.

The main point of this paper is that one should be very careful when formulating and using models with location. The intricacies involved in using location variables seem subtle and complex. The alternative mode of modeling spatial economies developed by the authors in separate papers (Berliant (1985) and Berliant and ten Raa (1986)) is not just a theoretical refinement of the canonical model of location theory, but generates dissimilar equilibrium results. Thus, it might be said that the two modes are qualitatively different.

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