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ABSTRACT

This paper examines the welfare theorems in the context of urban economics. The standard model of urban economics, which involves a continuum of agents located in a continuous space, is first described. Next, examples are given where both potential theorems fail for variants of the standard model in which preferences depend explicitly on location. Namely, we point out that a Pareto-optimum may not be an equilibrium even though preferences are continuous, convex and locally nonsatiated; and that an equilibrium may not be Pareto-optimal even though preferences are continuous and locally nonsatiated. A reason for this failure might be found in the heavy requirements that the model imposes on the equilibrium price of land. Firstly, it must fully capture the hedonic pricing of location while, at the same time, it must prevent the movement of consumers between locations. Secondly, at each location, it must equal the marginal rate of substitution between land and the numeraire commodity. Our examples might be relevant to other spatial models, such as differentiated product and hedonic models, location theory models and models of political party competition.

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1. INTRODUCTION

Ever since Aumann (1964), models with a continuum of agents, or commodities, or both, have been well-established in economics. The main reason for using such models are mathematical ease and the need to make precise the hypotheses of perfect competition. Continuum models are generally justified by demonstrating that they are close to models with a large, but finite, number of agents in terms of equilibria, comparative statics and welfare properties. Under these circumstances, the mathematical simplicity of the continuum model prevails and one may speak about an ideal form of perfect competition.

The focus of this paper is on a class of continuum models in urban economics. The field, rooted on the ideas of von Thünen (1826) and Haig (1926), has recently undergone considerable development. This began with the observation of Isard (1956, Chapter 8) that von Thünen can possibly be re-interpreted in the context of urban land-use, soon to be justified by a seminal presentation of Beckmann (1957). Beyond Mohring (1961) and Wingo (1961), who marked a conceptual transition between the earlier work of land economists and modern treatments of the subject, Muth (1961) and Alonso (1964) developed the foundations of what is now taken to be the paradigm of urban economics. In the simplest case, the model involves a density of consumers, identical with respect to income and tastes, distributed around a single centre which is located amidst a large, otherwise perfectly homogeneous area. Preferences are defined over the consumption of a composite good, which is found at the centre, and land. Land, in this case, can be thought of as a commodity continuum. Each consumer must occupy land at one and only one location, a requirement that yields a well-known nonconvexity in the related consumption possibility sets. Since the disposable income of consumers varies with distance from the centre due to differences in transportation costs, so does consumption. In equilibrium, consumption and prices are such that everyone enjoys the same utility level. The comparative statics behaviour of this, standard, model, in which the utility of consumers does not depend explicitly on location, has been discussed by Wheaton (1974) and, recently, by Pines and Sadka (1986). On the other hand, some welfare properties of a more general case, where the utility of consumers explicitly depends on location, have been first explored in a well-known paper of Mirrlees (1972).[1] He observed that the optimal allocation corresponding to a Benthamite social welfare function implies what has been termed "unequal treatment of equals", that is, different optimal levels of utility for identical consumers located at different distances from the centre.[2] This unusual property also holds in the case of the standard model.

The paradigm has been extended in several directions and it has provided us with valuable intuition concerning city structure: the associated qualitative explanations of urban phenomena stand often remarkably well against empirical evidence. Almost without exception, these variations involve a continuous population density distributed over a Euclidean space. In this respect, they belong to the growing tradition of continuum models in economic theory. Nevertheless, as Berliant (1985) has shown, they differ in that the usual justification for continuum models in economic theory does not apply to the corresponding spatial models.[3]

The existence of equilibrium for spatial continuum models has not been examined in great depth. It is known that, under certain conditions, an equilibrium exists for the standard model.[4] However, in the case where the utility of consumers explicitly depends on location, Berliant and ten Raa (1987) have constructed examples in which all of the classical conditions, such as continuity and convexity of preferences, hold but no equilibrium exists. In urban economics, this case arises either through the disutility of travel time to the centre (Alonso (1964)), or through an uneven distribution of environmental quality over the urban area. Examples of the latter case include major urban externalities such as traffic congestion (Solow (1972)), industrial pollution (Stull (1974)) and racial prejudice (Rose- Ackermann (1975)), local public goods (Schuler (1974)), spatially distributed amenities (Polinsky and Shavell (1976)), and the like. Since a large number of urban phenomena involve an uneven distribution of environmental quality, existence problems in the field may not be discounted as trivial.

The point of our paper is that, further to existence problems, both welfare theorems may not apply when the utility of consumers depends on location. In particular, we find examples in which the equilibria of this model, when they exist, are not necessarily Pareto-optimal even under the assumptions normally used to prove the first welfare theorem --continuity and local nonsatiation of preferences. Moreover, not all Pareto optima can be supported by prices, even when preferences are continuous, convex and locally nonsatiated. Hence the second welfare theorem can also fail under classical assumptions. It follows that, when the utility of consumers depends on location, the continuous model of urban economics does not exhibit the usual existence and welfare properties of other standard economic models. A reason for this might be found in the heavy requirements that the model imposes on the equilibrium price of land. Firstly, when location is treated --at least to some degree-- as a commodity, the hedonic pricing must be fully captured by the rent on land which, at the same time, must prevent the movement of consumers between locations. Secondly, at each location, the rent on land must be equal to the marginal rate of substitution between land and the composite good. One might expect cases in which any spatial distribution of rent fails to satisfy all these requirements simultaneously. Our examples confirm this intuition.

Section two displays our version of a standard model, including an explicit description of absentee landlords, which represents the typical method used in urban economics for closing the model. In section three, we provide an example where a Pareto-optimum does not have price-support. In section four, we provide an example where an equilibrium is not Pareto-optimal. Our examples are not exactly classical. In both cases, further to introducing an explicit preference for location, we find it necessary to modify some other assumption of the standard model. However, in no case do our particular deviations from the paradigm appear to represent general preconditions for a failure of the welfare theorems. Thus, when the utility of consumers explicitly depends on location, we cannot exclude the possibility that there may be closer replicas of the paradigm in which the welfare theorems do fail. We conclude the paper in section five with some general remarks.

2. A STANDARD MODEL OF URBAN ECONOMICS

Although there are several variants, the following description seems to represent what could be considered as a standard model of urban economics. There is a city on the Euclidean plane of radius b, with centre located at (0,0). The total urban area is πb^2. The total urban population is given by N, a positive real number, and consists of consumers identical with respect to their tastes and initial endowment. The location of a consumer in the city is indexed by the corresponding distance from the centre, $t \in [0,b] \equiv T$. Since orientation is irrelevant, the city obeys rotational symmetry. The land available at distance t is $2\pi t$. At a given location, a consumer has the non-negative orthant of $\mathbb{R}^2$, $\mathbb{R}^2_+$, for his consumption set. The two commodities are land, which is assumed to be homogeneous across locations, and a composite consumption good. All individuals receive an endowment of $e>0$ units of composite good, but no land. Let the quantity of composite good consumption and of land consumption be given by x and q respectively. For any consumer, preferences are represented by a utility function $u[x,q]$ which is continuous, strictly increasing and strictly quasi-concave. The corresponding indifference curves are smooth. Moreover, x and q are normal and their marginal rate of substitution, MRS, is such that $\ell im_{q \to \infty} MRS[x,q]=0$ and $\ell im_{x \to 0} MRS[x,q]=\infty$ --the Inada (1963) condition. These restrictions, which characterize a so-called "well-behaved" utility function, ensure uniqueness of the optimal choice and assume away corner solutions.

Consumers can consume land only at the place where they choose to locate. The composite consumption good serves as the numéraire commodity. The price of land, r, and the cost of transportation, k, vary with distance from the centre. Both are taken parametrically by the consumers. While the price of land is to be determined at equilibrium, the cost of transportation is a differentiable, strictly increasing function of distance from the centre.[5] Under these circumstances, the maximized utility of consumers can be represented by

$$v[r[t],e-k[t]] \equiv \underset{x,q,t}{\text{maximum}}\{u[x,q] \mid x+r[t]q=e-k[t];x\geq 0,q\geq 0\} \qquad (1)$$

for $t \in T$. Identical, strictly quasi-concave utility functions and positive, identical endowments imply that the optimal consumption choices $x[t]$ and $q[t]$ of consumers located at t are identical. Well-behaved utility implies that the corresponding quantities are strictly positive.

Given that the endowments of all consumers are equal and their preferences are identical, it is impossible to develop gains to trade among consumers. Therefore, unless we admit more than one type of consumer, it is only in closing the model that an economic situation can be produced. Toward this end, it is common in urban economics to introduce the concept of ***absentee landlords.*** At a given distance from the centre, an absentee landlord has the non-negative orthant of $\mathbb{R}$, $\mathbb{R}_+$, for his consumption set. His preferences are represented by a continuous, monotone increasing utility function $U[X]$, where X denotes the consumption of the composite good. Thus absentee landlords, unlike consumers, do not consume urban land. In the simplest case, at each distance t, there is a single absentee landlord who is endowed with all the land at t, $2\pi t$. Since every landlord controls only an insignificant portion of the urban land, he cannot exert any market power in his transactions with the consumers, because arbitrarily close substitutes are available in adjacent rings. Thus landlords take the price of land parametrically, as consumers do. Under these circumstances, the maximized utility of an absentee landlord at $t \in T$ can be represented by

$$V[r[t],t] \equiv \underset{X}{\text{maximum}}\ \{U[X] \mid X=2\pi t r[t];X\geq 0\}. \qquad (2)$$

Clearly, the only choice available to an absentee landlord at $t \in T$ is offering his land to the highest bidder in order to obtain as much X[t] as possible. In essence, the landlord at t compares the price of urban land r[t] with the opportunity cost of land.

The fraction of the urban population N located on the ring [t,t+dt] is given by n[t]dt. There is no idle land. Equality between demand and supply of urban land at distance t from the centre implies $n[t]q[t]=2\pi t$. Thus, the material balance conditions for land and the composite good can be written as

$$\int_0^b \frac{2\pi t}{q[t]} dt = N \tag{3}$$

$$\int_0^b \frac{2\pi t}{q[t]} (x[t]+k[t])dt = Ne - \int_0^b X[t]dt, \tag{4}$$

respectively. In equilibrium, the utility level attained by consumers is the same everywhere:

$$u[x[t],q[t]] = v^{\circ} \tag{5}$$

for $t \in T$. Finally, given that the opportunity cost of land, $\bar{r}$, is a positive constant, utility maximization by the absentee landlords requires

$$\min_{t \leq b} r[t] \geq \bar{r} \text{ a.s.} \tag{6}$$

for $t \in T$ in equilibrium.[6] Using the Walras law, one among the budget constraints in problems (1)-(2) and equations (3)-(5) is redundant if the marginal utility of income and the marginal cost of adding a consumer to the economy are positive.

An *allocation* is a list of non-negative measurable functions which specify the quantities of goods consumed by each agent of each type at each distance $t \in T$: $\{x[t],q[t];X[t]\}$; x,q and X measurable; $x \geq 0, q \geq 0, X \geq 0$ a.s.[7] A *feasible allocation* is an allocation satisfying the material balance conditions (3) and (4). A *competitive spatial equilibrium* is a feasible allocation, a distance $b>0$, a utility level v^{o} and a price system $r:T \to \mathbb{R}$, $r \geq 0$ a.s. and measurable such that, for almost every $t \in T$: (i) $(x[t],q[t])$ solves problem (1); (ii) $X[t]$ solves problem (2); and (iii) equations (5) and (6) hold. An *equilibrium allocation* is an allocation associated with a competitive spatial equilibrium for some price system. An *equilibrium relative to a price system* is a feasible allocation, a distance $b>0$ and a price system $r:T \to \mathbb{R}$, $r \geq 0$ a.s. and measurable, such that, for almost every $t \in T$: (i) $(x[t],q[t])$ maximizes $u[q,x]$ subject to $r[t]q+x \leq r[t]q[t]+x[t]$ and $x \geq 0, q \geq 0$; (ii) equation (6) holds.[8] Finally, a feasible allocation $\{A[t]\} \equiv \{x[t],q[t];X[t]\}$ is *Pareto-optimal* if there is no other feasible allocation $\{A'[t]\} \equiv \{x'[t],q'[t]; X'[t]\}$ such that: (i) $u|_{A'[t']} \geq u|_{A[t]}$ for almost all $t,t' \in T$, and (ii) $U|_{A'[t]} \geq U|_{A[t]}$ for almost all $t \in T$; with strict inequality holding for some set of positive measure. Thus, in a spatial context, a Pareto-optimum is a feasible allocation such that no reallocation of initial resources can improve the utility levels of landlords and/or consumers on a set of positive measure, leaving others at least as well-off.[9]

3. A PARETO-OPTIMAL ALLOCATION WHICH DOES NOT HAVE PRICE SUPPORT

In this section, we describe a Pareto-optimal allocation of the extended model which does not have price support. We begin our example assuming that consumers are freely mobile, so that the cost of transportation vanishes. We then adjust this example to include the cost of transportation.

Example 1. Let $u[x,q,t]=\min\{tx,q\}$ for $t \in T$. That is, consumers dislike locating near the centre. For fixed t, this utility is quasi-concave and increasing in x and q together. Since the absentee landlords consume only composite good, they become irrelevant to the price-support problem. We may, therefore, disregard them completely in this example. In order to have all initial resources fixed, let the border of the city be exogenous. Let the aggregate endowment of the composite good available to the consumers for net consumption equal $2\pi b$. Assuming that consumers at the same distance from the centre enjoy identical consumption bundles as in the standard model, (4) reduces to

$$\int_0^b \frac{2\pi t}{q[t]} x[t]dt = 2\pi b. \tag{7}$$

We now claim that the allocation

$$(x^*[t],q^*[t]) = \left(\frac{\pi b^2}{Nt}, \frac{\pi b^2}{N}\right) \tag{8}$$

for $t \in T$ is Pareto-optimal. It is trivial to show that $(x^*[t],q^*[t])$ satisfies (3) and (4), hence it is feasible. Moreover, since x and q are non-substitutable, there is no other feasible allocation (generated either by reallocating goods or relocating consumers) that improves upon $(x^*[t],q^*[t])$. This establishes the claim. At the Pareto-optimum, the utility level attained by the consumers is given by

$$u[x^*[t],q^*[t],t] = \min\{tx^*[t],q^*[t]\} = \frac{\pi b^2}{N}, \tag{9}$$

a constant across locations. Suppose there is a supporting price $r^*[t]$ for the optimum. Constant utility and decentralization imply that, under r^*, the value of $(x^*[t],q^*[t])$, say y, must also be a constant across locations in order to prevent movement of consumers. That is, $x^*[t]+r^*[t]q^*[t]=y$ or, equivalently,

$$r^*[t] = \frac{Ny}{\pi b^2} - \frac{1}{t} \tag{10}$$

which, for locations sufficiently close to the centre, is negative. So r^* is not a supporting price. In consequence, there is no price system relative to which this Pareto-optimum is an equilibrium.

The failure of the second welfare theorem in this example is caused by a price space which is not large enough. Clearly, the rent function cannot account for the location attribute and, at the same time, be equal to the (non-negative) marginal rate of substitution at each location while preventing movement between locations. If one allows more freedom in prices, it is possible to re-establish the second welfare theorem. Namely, if the price of the composite good can vary across locations as p[t] then it is easy to show that the price system

$$(p^*[t],r^*[t]) = \left(\frac{Nyt}{2\pi b^2}, \frac{Ny}{2\pi b^2}\right) \tag{11}$$

for $t \in T$ supports the Pareto-optimal allocation $(x^*[t],q^*[t])$. In this case, the numéraire is represented by the composite good at distance $2\pi b^2/Ny$. Variation in the price of the composite good over space can be accounted for by recognizing that transportation is costly. For example, make any quantity of the composite good available to consumers with income y free of charge at the centre, but impose the condition that consumers must transport this good to their location in order to consume it. Let the transportation rate per unit of the composite good be fixed. If it happens to equal $Ny/2\pi b^2$, (11) obtains. However, in any other case, $(x^*[t],q^*[t])$ is not an equilibrium.

The same conclusions apply when, as in the standard model, the cost of transportation is independent of the quantity transported. Suppose that $k[t]=t$. Then $x+(b-t)$ can be interpreted as the net consumption of someone at distance t from the centre. Taking this into account, we can write $u[x,q,t]=\min\{t(x+(b-t)),q\}$. We now let the aggregate endowment of the composite good available to the consumers for consumption equal $2\pi b+Nb/3$, and modify (7) accordingly. The Pareto-optimal allocation (8) becomes

$$(x^*[t],q^*[t]) = \left(\frac{\pi b^2}{Nt} - (b-t), \frac{\pi b^2}{N}\right) \tag{12}$$

for $t \in T$, which can be established using the same proof as before. At the Pareto-optimum, the utility level attained by the consumers is given once more by (9). Hence, if there is a supporting price $r^*[t]$ for the optimum, it must satisfy $x^*[t]+r^*[t]q^*[t]=y$ or, equivalently,

$$r^*[t] = \frac{Ny}{\pi b^2} - \frac{1}{t} + \frac{N(b-t)}{\pi b^2} \qquad (13)$$

which, for locations sufficiently close to the centre, is negative. It follows that, once again, r^* is not a supporting price.

4. AN EQUILIBRIUM ALLOCATION WHICH IS NOT PARETO-OPTIMAL

In order to produce an equilibrium allocation which is not Pareto-optimal, we found it necessary to construct a model involving two distinct types of consumer, $s\in\{1,2\}\equiv S$, and which allows explicitly for landless consumers in equilibrium. The total urban population of type s is N_s, a positive real number. Consumer utility is denoted $u_s[x,q,t]$. The maximized utility of consumers is now given by

$$v_s[r[t],e]] \equiv \underset{x,q,t}{\text{maximum}}\{u_s[x,q,t] \mid x+r[t]q\leq e; x\geq 0, q\geq 0\} \qquad (14)$$

for $s\in S$ and $t\in T$ where, as in the example of the previous section, the transport cost can be accounted for indirectly within the utility function itself. The solution to this problem determines $x_s[t]$ and $q_s[t]$. For fixed s, the density $f_s[t]$ represents the fraction of the total land at distance t from the centre which is occupied by consumers of type s. If type s does not live at t, $(x_s[t],q_s[t])=(0,0)$. Since some consumers of type s may decide to purchase no land but still locate at t, $\tilde{n}_s[t]$ is defined to be the density of such consumers. Thus $f_s[t]2\pi t/q_s[t]$ measures the density of type s consumers occupying land at distance t from the centre. In consequence, the material balance conditions become

$$\int_0^b f_s[t]\,\frac{2\pi t}{q_s[t]}\,dt = N_s - \int_0^b \tilde{n}_s[t]dt \qquad (15)$$

$$\int_0^b f_s[t]\,\frac{2\pi t}{q_s[t]}\,x_s[t]dt = \left[N - \sum_{s\in S}\int_0^b \tilde{n}_s[t]dt\right]e - \int_0^b X[t]dt, \qquad (16)$$

where $N \equiv N_1+N_2$. Finally, (5) is modified as

$$u_s[x_s[t],q_s[t],t] = v_s^o \qquad (17)$$

for $t\in T$ when type s is located at t.

An *allocation* here is a list of non-negative measurable functions which specify the quantities of goods consumed by each agent of each type, as well as the fraction of land consumed by consumers of each type, at each distance $t\in T$: $\{x_s[t],q_s[t],f_s[t],\tilde{n}_s[t];X[t]\}$; $x_s,q_s,f_s,\tilde{n}_s$ and X measurable; $x_s\geq 0, q_s\geq 0, f_s\geq 0, \tilde{n}_s\geq 0$ and $X\geq 0$ a.s. A *feasible allocation* is an allocation satisfying the material balance conditions (15) and (16) such that, for $s\in S$ and $t\in T$, $0\leq f_s[t]\leq 1$ and $\sum_{s\in S} f_s[t]=1$ a.s.(t). A *competitive spatial equilibrium* is a feasible allocation, utility levels v_s^o and a price system $r:T\to\mathbb{R}$, $r\geq 0$ a.s. and measurable such that: (i) for each $s\in S$ and almost every $t\in T$ with $f_s[t]>0$, $(x_s[t],q_s[t])$ solves problem (14); (ii) for almost every $t\in T$, X[t] solves problem (2); and (iii) for each $s\in S$ and almost every $t\in T$, equations (6) and (17) hold. Finally, a feasible allocation $\{A_s[t]\}\equiv\{x_s[t],q_s[t],f_s[t],\tilde{n}_s[t];X[t]\}$ is *Pareto-optimal* if there is no other feasible allocation $\{A'_s[t]\} \equiv \{x'_s[t], q'_s[t], f'_s[t], \tilde{n}'_s[t]; X'[t]\}$ such that: (i) $u_s|_{A'[t']} \geq u_s|_{A[t]}$ for all $s \in S$, and almost every $t, t' \in T$ with $f_s[t], f'_s[t'] > 0$ or $\tilde{n}_s[t], \tilde{n}'_s[t'] > 0$, and (ii) $U|_{A'[t]} \geq U|_{A[t]}$ for almost every $t \in T$; with strict inequality holding for some positive measure.

Example 2. When consumers are freely mobile, let their utilities be given by $u_1[x,q,t] = \min\{x, (1+t)q+1-t\}$ and $u_2[x,q,t] = x + (\min\{(1+t)/t, 3\})q$. Each consumer is endowed with two units of the composite good and no land. To simplify calculations, let $N_1 = N_2 = 3\pi$, $\bar{r} = 2$ and $b = 1$. In equilibrium, if type one consumers live at t,

$$x_1[t] = (1+t)q_1[t]+1-t = u_1[x_1[t], q_1[t], t]. \tag{18}$$

Since, according to (17), $u_1[x_1[t], q_1[t], t] = u_1[x_1[0], q_1[0], 0] \equiv u_1[0]$ in equilibrium, (18) implies

$$q_1[t] = \frac{u_1[0]-1+t}{1+t}. \tag{19}$$

Furthermore, since $q_1[0]=0$, $u_1[0]=1$ in equilibrium. Thus, using (18) and (19),

$$(x_1[t], q_1[t]) = \left(1\ ,\ \frac{t}{1+t}\right). \tag{20}$$

Since the corresponding budget constraint is given by $x_1[t]+r[t]q_1[t]=2$, it follows that

$$r[t] = \frac{1+t}{t}. \tag{21}$$

Using (21), for consumers of type two, $u_2[x,q,t]=x+3q$ when $t<1/2$ and $u_2[x,q,t]=x+r[t]q$ when $t\geq 1/2$. Therefore,

$$r[t] >(=) \left(\frac{\partial u_2}{\partial q}/\frac{\partial u_2}{\partial x}\right) \text{ for } t<(\geq) \frac{1}{2}, \tag{22}$$

which implies that consumers of type two can maximize utility only for $t\in[1/2,1]$. In this area, there are multiple solutions to their problem.

We choose to study an equilibrium where consumers of type two consume only their initial endowment --two units of composite good, and where they are evenly distributed over $[1/2,1]$, $\tilde{n}_2[t]=6\pi$ for $t\in[1/2,1]$. In this equilibrium, since type one consumers locate everywhere in T, landlords collect $2\pi tr[t]=2\pi(1+t)$ for $t\in T$.[10] Thus the equilibrium is given by $\{x_1[t],q_1[t],f_1[t],\tilde{n}_1[t];X[t]\}=\{1,t/(1+t),1,0;2\pi(1+t)\}$ and $r[t]=(1+t)/t$ for $t\in T$, $\{x_2[t],q_2[t],f_2[t],\tilde{n}_2[t]\}=\{2,0,0,6\pi\}$ for $t\in[1/2,1]$ and $(v_1^o,v_2^o)=(1,2)$. We now argue that this equilibrium allocation can be Pareto-dominated by another feasible allocation, constructed by moving some positive mass of type one consumers from any set of $t>1/2$ to the centre. All they take with them is their one unit of numéraire that they possess in equilibrium. After the move, they consume no land, which is consistent with zero land supply at the centre. However, their utility remains the same as in equilibrium. The utility of the landlord at t is also unchanged, as he retains the same amount of numéraire paid to him in equilibrium. Since the land occupied in equilibrium by the type one consumers who have been moved is now free, if it is given to the type two consumers who reside at t in equilibrium, the utility of the latter increases. Thus the new allocation Pareto-dominates the equilibrium allocation.

The failure of the first welfare theorem in this example occurs because rent is forced to become infinite at the centre in order to prevent movement and, at the same time, provide a utility maximum subject to the budget at $t=0$.[11] Under these circumstances, type one consumers are indifferent between staying at $t>1/2$ with land or moving to the centre with none. Although the second alternative is cheaper, any positive amount of land at the centre would break the budget constraint. On the other hand, although they can consume their additional unit of initial endowment if they live at $t=0$ in the equilibrium, this would not improve their condition since their preferences are not strictly monotone in x when $q=0$.[12]

When consumers are not freely mobile, let their utilities be modified as $u_1[x,q,t]=\min\{x+(1-t),(1+t)q+2(1-t)\}$ and $u_2[x,q,t]=x+(1-t)+(\min\{(1-t^2)/t,3/2\})q-(1-t)$. For $b=1$, $x+(1-t)$ represents consumption net of transport costs. As in the previous case, each consumer is endowed with two units of the numéraire and, to simplify calculations, $N_1=N_2=4\pi/3$ and $\bar{r}=0$. Proceeding exactly as before, we find that

$$(x_1[t],q_1[t]) = \left(1+t\ ,\ \frac{2t}{1+t}\right) \tag{23}$$

which implies

$$r[t] = \frac{1-t^2}{t} \tag{24}$$

since the corresponding budget constraint is given by $x_1[t]+1+r[t]q_1[t]=2$, where $x_1[t]+1$ is the total amount of the numéraire left after purchasing land. For consumers of type two, (22) holds. If we choose the equilibrium where they consume only composite good and they are evenly distributed over [1/2,1], $\tilde{n}_2[t]=8\pi/3$ for $t\in[1/2,1]$, and a Pareto improvement can be obtained by moving type one consumers to the centre and giving their land to consumers of type two.

5. CONCLUDING REMARKS

Pareto-optimality in urban economics has typically been confined to social welfare maximization. Since the social welfare function takes the form $\int n[t]w[u[x[t],q[t],t]]dt$, where w is a weighting function, Pareto-optimal allocations which maximize social welfare must be such that consumers at the same distance from the centre have the same consumption. A sufficient condition for this further requirement is strict concavity of the composition of the weighting function and the utility function with respect to x and q.[13] For utilitarian optima, where w is the identity function, the requirement of strict concavity passes to the utility function itself. Since our utility functions are only concave in x and q, they will be improper for some utilitarian optima. However, in more general cases where w is not the identity function, as, for example, under a positive degree of aversion to inequality, concave utilities can be entirely consistent with the composition of w and u strictly concave in x and q.

Hartwick (1982) has shown that strict concavity of w with respect to x and q does not rule out pathological cases in the context of social welfare maximization. We believe that this happens because either the objective $n[t]w[\cdot]$ might not be concave in all variables, or the constraint set defined by the feasibility conditions might not be convex, or both. On a more general level, strict concavity has little to do with the welfare theorems: it is a cardinal property, and utility aggregation never occurs in either the standard proofs or in our examples. Furthermore, socially optimal allocations are only a subset of Pareto-optimal allocations. Thus, even though our examples may not apply to some cases of social welfare optimization, they are still relevant in the more general context of welfare economics.

A reformulation of the examples for a finite or countable number of locations is quite possible and, in fact, weakens both the continuity restriction on the utility functions as well as the positive measure requirement used in the definition of a Pareto optimum. In consequence, the commodity continuum generated by urban land on the Euclidean plane does not appear to be the reason behind the failure of the welfare theorems.

Suzanne Scotchmer has rightly pointed out that the driving force in example two is represented by consumers who do not need to occupy land at $t=0$. Thus one might be tempted to believe that a necessary condition for the validity of the welfare theorems is that every consumer requires a positive amount of land. Since there is no analogous restriction in exchange economies where the land nonconvexity is absent, such claim would be of interest. However, although it is true that this applies to our second example, there can be counter-examples otherwise, as suggested by our first example. Furthermore, neither $q[t]>0$ for all $t\in T$, nor strict monotonicity of preferences in each good, which has been violated in our example two, are generally used as either necessary or sufficient conditions for proofs of the welfare theorems --even in the infinite- dimensional setting. The conditions normally assumed for the first welfare theorem are continuity and local nonsatiation of preferences. The preferences of a type one consumer in example two are locally nonsatiated both for fixed location or allowing location to vary. For the second welfare theorem, convexity of preferences is normally added to the previous conditions. All of these are satisfied in example one.

Our examples might be relevant to other spatial models on the demand side of an economy, such as models of product differentiation, by interpreting the locational attribute of urban economics as a general hedonic attribute or quality. However, the reinterpreted model is different from typical product differentiation models, such as Mas-Colell (1975) and Jones (1984), in which commodities can be purchased in more than one location simultaneously. On the other hand, the supply side of a spatial economy includes models of product differentiation such as Novshek (1980), and location theory models in the great tradition of Hotelling (1929) together with their derivatives (see, for example Kramer (1977)). In all these models, the demand continuum is passive. Nevertheless, in more general cases where both consumers and producers can choose their location, or where supply itself is represented by an active density of agents, our examples may well become relevant.

FOOTNOTES

1 Mirrlees (1972) pertains to the second welfare theorem. The first welfare theorem, on the other hand, has been studied by Scotchmer (1985) in a general equilibrium context.

2 Subsequent work, including Riley (1973) and Dixit (1973), established that all symmetrical social welfare functions, other than the Rawlsian, can imply unequal treatment of equals at the optimum. As Wildasin (1986) points out, this happens when the marginal utility of income varies with location in equilibrium. Under these circumstances, social welfare can be improved through a transfer of resources toward consumers with higher marginal utility of income.

3 Berliant (1985) shows that a continuum of agents, each holding a positive area of land in a Euclidean space is impossible; and that any sequence of economies, each with a finite number of agents, tending to a limiting economy with a continuum of agents, has the property that the land holdings and endowments of agents must tend to zero on average. This contrasts with, say, Hildenbrand (1974) where average endowments and consumption are *positive*, but the *fraction* of total commodities consumed by an agent tends to zero. Thus the continuous model of urban economics cannot be interpreted as an approximation to a large, finite economy in the standard sense exemplified by Hildenbrand (1974).

4 The existence of an equilibrium for the standard continuum model has been established in Fujita and Smith (1986), given some restrictions on preferences and demand which are stronger than those described in our section two. On the other hand, Schweizer, Varaiya and Hartwick (1976), Ginsburgh, Papageorgiou and Thisse (1985) and Scotchmer (1985) deal with the same subject for a discrete set of locations.

5 The cost of transportation arises because consumers must visit the centre regularly. This can be justified if, for example, the endowment of every consumer is to be obtained from the centre at regular time-intervals.

6 Note that, when $\bar{r}$ is fixed, we are dealing with a partial equilibrium concept. The equilibrium price of land is given by

$$r^{o}[v^{o},e-k[t]] \equiv \underset{x,q}{\text{maximum}} \{(e-k[t]-x)/q \mid u[x,q]=v^{o}; x\geq 0, q\geq 0\}$$

for $t\in T$. Since the utility and transportation cost functions are continuous, so is $r^{o}[\cdot]$. Therefore, $r[b]>\bar{r}$ in equilibrium implies the existence of a sufficiently small $\epsilon>0$ such that $r[b+\epsilon]>\bar{r}$. Outside of the urban area, absentee landlords face a price of land equal to its opportunity cost. However, at $b+\epsilon$, consumers are prepared to pay a price for land which is greater than its opportunity cost. It follows that the absentee landlord at $b+\epsilon$ does not maximize utility --a contradiction.

7 Implicitly, when we refer to measure, we mean the Lebesgue measure on T.

8 Unlike the definition of a competitive equilibrium, this definition allows consumers to begin with an endowment of land.

9 Introducing a space of agents is notationally complex. Since a more precise notation does not seem to further our understanding of the problem at hand, we use these intuitive definitions for the sake of simplicity.

10 The population condition (15) becomes

$$\int_0^1 \frac{2\pi t}{q_1[t]}\, dt = \int_0^1 2\pi(1+t)dt = (\pi t^2+2\pi t)\Big|_0^1 = 3\pi$$

$$0 = 3\pi - \int_{1/2}^1 6\pi dt = 3\pi-3\pi$$

for types one and two respectively. On the other hand, the composite good condition (16) becomes

$$\int_0^1 2\pi(1+t)dt = (6\pi-3\pi)2 - \int_0^1 2\pi(1+t)dt.$$

11 In this way, the failure of the first welfare theorem here is similar to that in some overlapping generations models.

12 These comments belong to James Mirrlees and Suzanne Scotchmer.

13 For a particular distance from the centre,

$$w[u[\lambda(x_1,q_1)+(1-\lambda)(x_2,q_2)]] > \lambda w[u[x_1,q_1]]+(1-\lambda)w[u[x_2,q_2]]$$

holds under strict concavity, where $0<\lambda<1$ and the subscripts specify different consumption bundles. For $\lambda=1/2$, we have

$$2w\left[u\left[\frac{x_1+x_2}{2}, \frac{q_1+q_2}{2}\right]\right] > w[u[x_1,q_1]]+w[u[x_2,q_2]],$$

which implies that when consumers at a particular distance t from the centre are treated identically, the sum of socially evaluated utilities at t is maximized.

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