Rochester Center for

Economic Research

Supply and Equilibrium in an Economy with Land and Production

Berliant, Marcus and Hou-Wen Jeng

Working Paper No. 98
September 1987

University of

Rochester

SUPPLY AND EQUILIBRIUM IN AN ECONOMY WITH
LAND AND PRODUCTION

Marcus Berliant

and

Hou-Wen Jeng*

Working Paper No. 98

September 1987

*Department of Economics, University of Rochester, Rochester, NY 14627.

The comments of John Boyd III and Ping Wang, and financial support provided by National Science Foundation Grant #SES-8605629 are gratefully acknowledged. The authors bear full responsibility for the contents.

Abstract

A model with a finite number of consumers trading in land that can be changed through production is examined. Unlike most other models in urban economics, land is modelled as measurable subsets of a compact set. Formally, there are three major results. Conditions under which a solution to the producer's profit maximization problem exists are found. Secondly, conditions under which a solution to the consumer's problem exists are found. The assumptions used to prove these two results are standard. Finally, the existence of an equilibrium is demonstrated.

I. Introduction

Continuum models, such as monocentric city models(Alonso[1], Beckmann[2]), are widely used in urban economics and regional science to explain the residential location of consumers. In general, they yield first-order conditions and comparative statics that have empirical relevance.

As noted in Wheaton [21], although the assumptions of the monocentric city model are simple and essential for mathematical tractability, they eliminate some important features of urban structure. A criticism is made by Berliant [4] on the consistency of the assumptions employed in the monocentric city model. As it is typical in these models to assume that the land of the economy is in a Euclidean space and there is an infinity of consumers, Berliant [4] shows that most agents would own a parcel of land of measure 0 because there is only a countable number of disjoint subsets of positive area in any partition of a Euclidean space. As a consequence, the equilibria and the comparative statics of monocentric city models are not necessarily close (in the Hildenbrand [16] sense) to any reasonable finite economies. Moreover, the existence of equilibrium and the classical welfare theorems have not been examined in great depth for the monocentric city models. Berliant and ten Raa [7] show that it is possible to construct examples such that all of the classical conditions of general equilibrium theory hold but no equilibrium exists. Berliant, Papageorgiou, and Wang [9] find examples where the first or second welfare theorems fail even under classical conditions.

Since the monocentric city models suffer from the mathematical problem and the theoretical difficulties mentioned above, the conclusions they reach are, therefore, open to question. In a series of papers by Berliant, an alternative way of modelling the commodity land such that it can be freely subdivided and recombined and such that each trader has positive measure of land has been proposed. In particular, land is modelled as a measurable subset L, of R^2, and the consumption set of each of a finite number of traders consists of the measurable subsets $\mathcal{B}$ of L. Any commodities which are indivisible and heterogeneous in the sense that the commodity would not be the same if it is divided, and such that different parts can have different things embedded in them, can also be modelled this way.

In Berliant [4] under an independence assumption, the utility function is shown to take the form:

$$\int_B h_i(x)dm(x) \tag{1}$$

where m is Lebesgue measure on land, $B \in \mathcal{B}$ is the parcel that the agent i owns, and $h_i > 0$ a.s. is integrable on land L. Since the utility from the union of two parcels is equal to sum of the utility of the two parcels, this linear-type utility function actually rules out complementarities between parcels[1], but is useful to obtain results. Assuming the price of land can be represented by the integral of a price density, a characterization of the demand for land with a linear utility function (1) is given in Berliant [3] and the existence of equilibrium, as well as the welfare theorems, are proved in Berliant [5] for an exchange economy with land. In an effort to generalize the utility function to a nonlinear one, Berliant and ten Raa [8] have proposed a topology on land parcels and showed that a solution to the consumer's problem will always exist for utilities continuous with respect to the topology. While this work can be viewed as an attempt to build a model of land closer to reality, only the properties of the pure exchange economy with land have so far been examined. The comparative statics of a complete model including production might be important and could be compared with those of the New Urban Economics.

A complete model could also have an impact on the local public goods literature.

In fact, the original motivation for this line of research on economies with land was to combine a model of land with a model of local public goods to see if this combination could reverse any of the negative results of Bewley [11] concerning local public goods. Clearly, production is necessary if one is to include local public goods in the model.

In this paper, we develop a general equilibrium model of land with production which integrates the markets for land and mobile goods. As will become clear later on, this model allows for a higher degree of differentiation in the commodity space than most other models in the infinite commodity literature. The same parcel of land with different production input-output densities on it are treated as different commodities. The analysis follows the spirit of the models mentioned above, using a set of assumptions that enables us to get a solution to the producer's and consumer's problems and the general properties of the equilibrium. In equilibrium, a producer and consumers jointly determine the production input-output density on land, and thus determine which differentiated products are produced and consumed.

In our framework, land can be purchased from the consumers and can be modified in the sense that a change in land through production (of housing, say) can increase or decrease the value of the land. Thus, the price of land depends on its area, shape, location, and anything embedded in it through production. Any production activity which changes the elements of land and limits its further use, such as strip mining, may reduce its value. Building a house on a piece of land can be considered adding value to the land since the price of land includes everything embedded in it.

The price system for land deserves special attention since land is productive and only the price of land can reflect the fact that land has been altered through production. In other words, an effective price system should be able to assign appropriate values to parcels of land with different input-output densities in addition to different areas, shapes and locations, as land parcels are marked by input-output density after production. Therefore, the input-output density enters the price system for land as an argument. This characteristic of land prices in the production economy distinguishes our model and analysis from other works in the infinite dimensional commodity literature since the land prices are densities on the product space of land L and R^k, where k is the number of mobile commodities. Note that duality theory can not be used here to get a price space for land as the commodity space, which contains $\mathcal{B}$ as a component, is not linear.

Although the complexity of the land price arises in a very natural way, it inevitably introduces some difficulties to the proof of the existence of solutions to the producer's and consumer's problems and to the proof of the existence of an equilibrium of the economy. First, it is not obvious what restrictions should be placed on the price space to get the convexity needed to apply a fixed point theorem. Some topology on input-output densities is required to define the continuity of the land price. Thus, in order to get a solution to the producer's problem we endow the commodity space and the spaces containing input-output densities and price systems with some topological structures with respect to which the profit function can be shown to be continuous. We also limit the price systems to a space which satisfies a convexity restriction in order to get the existence results.

Section II describes the economy, makes definitions and assumptions regarding the production set, and defines the producer's problem. In section III we demonstrate the existence of a solution to the producer's problem. The assumptions needed to guarantee existence of a solution to the producer's problem when the production set depends on the land used are explored.

In section IV, preferences and endowments of consumers are defined. We assume consumers have the choice of the production density on the land demanded by them. The mathematical tools used to establish that there is a solution to the consumer's problem include the theorems of Ascoli and Lyapunov. Next, in section V, we demonstrate the existence of an equilibrium, which is obtained by embedding the demand for land in nonnegative measures and using the Ky Fan fixed point theorem to get a "mixed equilibrium" in which consumers can buy fractions of parcels. Then we show that some mixed equilibrium thus obtained is actually a competitive equilibrium. Finally, some concluding remarks and possible extensions are offered in section VI.

II. The Economy

Consider an economy with k mobile goods and land L, a compact subset of R^n. Let $\mathcal{B}$ denote the σ-algebra of measurable sets in L. Consider one firm in the economy using land and other factors in the production of k mobile goods and any other goods that are immobile, such as housing, on the land. For this firm, the production decision is to choose a production plan such that profit is maximized.

We formally define the space of Lipschitz functions, the production set Y, and the price system P for land as follows:

Definition 1: *S_1 is the space of all k-dimensional Lipschitz functions defined on L with constant $c_1 > 0$. $b \in S_1$ if and only if $b = 0$ on the boundary of L and $\|b(x) - b(y)\|_k \leq c_1 \|x - y\|_n$ for x and $y \in L$ where $\|.\|_k$ is the k-dimensional Euclidean norm. S_1 is endowed with the uniform topology(Munkres [19] p.122) induced by the sup metric $sp(b_1, b_2) = \sup_{x \in L} \|b_1(x) - b_2(x)\|_k$ for $b_1, b_2 \in S_1$. $C^0(L)$ is the space of continuous functions on L. 1_B is the indicator function for land parcel $B \in \mathcal{B}$ where $1_B(x)$*

is 1 when $x \in B$, 0 otherwise. Define $B^c = \{x \in L \mid x \notin B\}$ and $B/C = \{x \in L \mid x \in B$ and $x \notin C\}$ for $B, C \in \mathcal{B}$.

b is the input-output density which stands for the production activity undertaken on land L, and we restrict b to the space of Lipschitz functions in order to obtain a solution to the firm's problem and an equilibrium of the economy. For $x \in L, b(x)$ is the density of inputs and outputs used at x. Outputs are given positive coordinates while inputs are given negative ones. This topology is chosen for S_1 because, as was discussed in the introduction, b is an argument in the land price system and the weak* topology on S_1 (embedding S_1 in L^∞) will not be strong enough to ensure the continuity of the land price in input-output density b.

Definition 2: *Let Ω denote the subset of $S_1 \times \mathcal{B} \times R^k$ such that if $(b, A, d) \in \Omega, b(x) = 0$ for any $x \notin A$. A production set Y is a subset of Ω such that for any $(b, A, y) \in Y, y = \int_A b(x)dm(x)$ where m is Lebesgue measure on land.*

$b(x) = 0$ for any $x \notin A$ means that the input-output density is only changed on A under the production plan (b, A, y). Since $b = (b_1, ..., b_i, ..., b_k)$, where b_i is an $L^1(L)$ function[2] mapping L to R,

$$y = \int_A b(x)dm(x) = (\int_A b_1(x)dm(x), ..., \int_A b_i(x)dm(x), ..., \int_A b_k(x)dm(x))$$

is the input-output vector on A. In the following definitions, we put some topological structure on the σ-algebra $\mathcal{B}$ needed for the proofs in sections III and IV.

Definition 3: *Let R_+ denote the set of non-negative real numbers. $S_2 \subseteq C^0(L \times R^k)$ is the set of Lipschitz functions mapping $L \times R^k$ to R_+ with constant $c_2 > 0$. Thus, $P \in S_2$ if and only if $P = 0$ on the boundary of L and*

$$\mid P(x, r_1) - P(y, r_2) \mid \leq c_2(\|(x, r_1) - (y, r_2)\|_{n+k})$$

for $x, y \in L$ and $r_1, r_2 \in R^k$. S_2 is endowed with the uniform topology.

Definition 4: *A price system for land is $P \in S_2$. Implicitly[3], $P: L \times S_1 \to R_+$ via $P(., b(.))$ if $b \in S_1$.*

Thus, land price can differ both by location and production activity.

Definition 5: *The set of price systems, PS, is $\{(P, q) \in S_2 \times R^k | \ q = (q_1, ..., q_k), q_j \geq 0 \forall j, \forall b \in S_1, \int_L P(x, b(x)) dm(x) + \sum_{j=1}^k q_j = 1\}$. We denote an element of PS by (P, q) where q is the price system for mobile goods. Notice that P prices production not undertaken.*

Assumption A1: *If $(b, A, y) \in Y$ and $\{x \in L | b(x) \neq 0\} \subseteq A'$, then $(b, A', y) \in Y$. If $(b_t, A_t, Y_t) \in Y \forall t$ and $b_t \to b$, then $\exists y \in R^k, A \in \mathcal{B}$ with $(b, A, y) \in Y$.*

The first part of the above assumption means that if a production plan is feasible, then the same plan using more or less land (through set inclusion) is also feasible provided that the parcel with non-zero density is included. Implicitly, a producer can artificially employ all land in his production plan since only the land parcel with non-zero input-output densities matters. The second part of A1 states that for any $\{b_t\}_{t=1}^{\infty} \in S_1$ which converges to b^* in the topology defined above, there exist $y^* \in R^k$ and $A^* \in \mathcal{B}$ such that $y^* = \int_{A^*} b^*(x) dm(x)$ and thus, production is feasible (there always is such a y^*). In other words, the production set is closed.

That Y is closed and the equicontinuity of a maximizing sequence $\{b_t\}_{t=1}^{\infty}$ are parts of the sufficient condition for the proof that there is a solution to the producer's problem.

The firm's problem is to solve the following maximization problem:

$$\max_{(b,A,y)\in Y} \int_A (P(x,b(x)) - P(x,0))dm(x) + q.y \tag{2}$$

where $(P,q) \in PS$ and $y \in R^k$ represent prices of all goods and quantities of the k mobile goods, respectively. The maximization problem specified above indicates that the firm uses land as both input and output and in its production of mobile goods. Ordinary production is captured when Y does not require land for production.

III. Existence of A Solution to The Producer's Problem

We will prove the existence of a solution to the producer's problem in theorem 1 below. The proof proceeds as follows. Take a maximizing sequence $\{(b_t, A_t, y_t)\}_{t=1}^{\infty}$ for the profit function. If (b_t, A_t, y_t) is an element of Y for every t, then $A_t \in \mathcal{B}$, $b_t \in S_1, y_t = \int_{A_t} b_t(x)dm(x) \in R^k$, and $b_t(x) = 0$ for any $x \notin A_t$. The main problem is to show that there exists $(b^*, A^*, y^*) \in Y \subseteq S_1 \times \mathcal{B} \times R^k$ such that $A_t \to A^*$ in the sense that the appropriate integrals converge, $b_t \to b^*$ uniformly and $y_t \to y^*$ coordinatewise, and (b^*, A^*, y^*) is the maximizer of the profit function.

Theorem 1: *If $Y \neq \phi$ is a production set satisfying A1, then there exists a solution to the producer's problem.*

Proof: It is a priori possible that supremum of the profit function is ∞, and consequently not bounded above. But this will not stop us from finding a maximal production plan. Let $\{(b_t, A_t, y_t)\}_{t=1}^{\infty} \in Y \subseteq S_1 \times \mathcal{B} \times R^k$ be a maximizing sequence for the profit function tending to the supremum, and let $\{x_j\}_{j=1}^{\infty}$ be a sequence in L such that $x_j \to x \in L$. Then since $\{b_t\}_{t=1}^{\infty}$ are in the space of Lipschitz functions for the constant $c_1 > 0, b_t(x_j) \to b_t(x)$ as $j \to \infty$ for all t, where the convergence does not depend on the choice of t, i.e., $\{b_t\}_{t=1}^{\infty}$ is equicontinuous. Since S_1 is closed in the sup metric, by Ascoli's theorem (Munkres [19] p.276), $\{b_t\}_{t=1}^{\infty}$ is contained in a compact set. Thus, there exists a subsequence $\{b_{t_l}\}_{l=1}^{\infty}$ converging to b^* in S_1, i.e., $b_{t_l}(x) \rightarrow b^*(x)$ uniformly for all $x \in L$. We pass to this subsequence, and we denote it by $\{b_t\}_{t=1}^{\infty}$.

Given L is compact and b^* is continuous, the image of b^* is bounded, so there exists a $\delta_1 > 0$ such that $\|b^*(x)\|_k < \delta_1$ for every $x \in L$. Note that as $b \in S_1$, for $\epsilon_1 > 0$, there exists a $T > 0$ such that for all $t > T$, and $x \in L, \|b_t(x)\|_k < \|b^*(x)\|_k + \epsilon_1$. As L is compact,

$$\int_L (b^*(x) + \epsilon_1) dm(x) \leq (\delta_1 + \epsilon_1) m(L) < \infty.$$

Since $\lim_{t\to\infty} b_t(x) = b^*(x)$ for $x \in L$, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty} \int_L b_t(x) dm(x) = \int_L b^*(x) dm(x).$$

Since $P(.,b(.))$ is continuous in both arguments and $\{b_t\}_{t=1}^{\infty}$ converges uniformly to b^*, we have $\lim_{t\to\infty} P(x, b_t(x)) = P(x, b^*(x))$ for every $x \in L$. Given L is compact, and P and b^* are continuous, the image of $P(x, b^*(x))$ is bounded, so there exists a $\delta_2 > 0$ such that $\mid P(x, b^*(x)) \mid \leq \delta_2$ for every $x \in L$. Also since $P \in S_2$ and $b \in S_1$, given $\epsilon_2 > 0$, there exists a $T > 0$ such that for all $t > T, P(x, b_t(x)) < P(x, b^*(x)) + \epsilon_2$ for every $x \in L$. Since L is compact,

$$\int_L [P(x, b^*(x)) + \epsilon_2] dm(x) \leq (\delta_2 + \epsilon_2) m(L) < \infty.$$

By Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty} \int_L P(x, b_t(x)) dm(x) = \int_L P(x, b^*(x)) dm(x).$$

Hence,

$$\lim_{t\to\infty} \int_L [P(x, b_t(x)) - P(x, 0) + q.b_t(x)] dm(x) = \int_L [P(x, b^*(x)) - P(x, 0) + q.b^*(x)] dm(x)$$

By A1, $(b_t, L, y_t) \in Y$ for all t, $(b^*, L, y^*) \in Y$ and (b^*, L, y^*) attains the supremum value[4] of the profit function.///

It is interesting to see what restrictions must be placed on the production set to guarantee a solution to the producer's problem if there is a feasible production set for each element of the σ-algebra. The dependence of the production set on land can be modelled as follows. There is a correspondence Λ from $\mathcal{B}$ to S_1 such that a production plan $(b, A, y) \in Y$ iff we have $b \in \Lambda(A), y = \int_A b(x)dm(x)$. The following corollary proves that there is a solution to the producer's problem.

Corollary 1: *For each $A \in \mathcal{B}$, let $\Lambda(A) \subseteq S_1$. Assume that if $\{x \in L | b(x) \neq 0\} \subseteq A$, then $b \in \Lambda(A)$, and if $b = lim_{t \to \infty} b_t$ with $b_t \in \Lambda(A_t)$ for some A_t, then $b \in \Lambda(A)$ for some A. Also assume that $0 \in \Lambda(A) \forall A$. Then there exists a solution to the producer's problem.*

Proof: If $Y \equiv \{(b, A, y) | b \in \Lambda(A), y = \int_A b(x)dm(x)\}$, then Y satisfies A1. Following theorem 1, there exists a solution to the producer's problem.///

Next we provide an example to illustrate the relationship between land and the production set. Suppose building a house needs some minimum amount of land: $m(A) \geq r$. $\Lambda(A)$ in this case is defined as 0 if $m(A) < r$ and $\bar{Y}$ if $m(A) \geq r$ where $0 \in \bar{Y} \subseteq S_1$. If $\bar{Y}$ is closed, then it represents an admissible production technology.

IV. The Existence of A Solution to The Consumer's Problem

In this section we will prove that there exists a solution to the consumer's problem. The proof uses techniques similar to those used in proving Theorem 1. There are N consumers in the economy(indexed by i and j) who maximize their utilities given a budget constraint. Consumers derive utility from consuming land and mobile goods. Implicitly the input-output density b affects the demand for land and mobile goods. We define the consumption set and endowment for each individual and make assumptions concerning consumer preferences as follows.

Definition 6: *For every consumer, the consumption set is* $X = \{(b, B, d) \in \Omega \mid d$ *is non-negative in every component* $\}$. *Without losing generality, consumer i's endowment is given by* $(0, E_i, e_i) \in X$ *such that* $\cup_{i=1}^{N} E_i = L$ *and* $E_i \cap E_j = \phi, i \neq j$, *and* $e_i \geq 0$ *in each component, strictly positive in some component.*

An element of the consumption set X specifies a land parcel, what is put on it, and consumption of mobile goods. $\cup_{i=1}^{N} E_i = L$ means all land is owned by someone, and $E_i \cap E_j = \phi, i \neq j$ means that two consumers cannot be endowed with parcels that overlap. As $X \subseteq \Omega$, consumer has zero demand for b on land parcels other than his own, which implies that there is no externality in the consumption of land.

Assumption A2: *The preferences of consumer i are represented by a utility function of the following form:*

$$U_i(b, B, d) = u_i\left(\int_B h_i(x, b(x))dm(x), d\right)$$

for $(b, B, d) \in X$ *where* h_i *is continuous on* $L \times R^k$, $h_i(x, 0) > 0 a.s.$ *and* $u_i\colon R_+^{k+1} \to R$ *is continuous.*

Preferences over land parcels depend on what is built on or mined from the parcels. Note that the utility function allows complementarities between mobile goods and land, but not between land parcels[5].

The consumer's problem is to solve the following maximization problem:

$$\max_{(b,B,d)\in X} u_i(\int_B h_i(x, b(x))dm(x), d) \tag{3}$$

$$s.t. \int_B P(x, b(x))dm(x) + q.d \leq \int_{E_i} P(x, 0)dm(x) + q.e_i$$

$$+\theta_i[\int_L (P(x, b_y(x)) - P(x, 0))dm(x) + q.y] \equiv I_i. \tag{4}$$

where $\sum_{i=1}^{N} \theta_i = 1, B \in \mathcal{B}$ and $q, e_i, d \in R_+^k$. The basic setup follows conventional general equilibrium theory where $\int_{E_i} P(x, 0)dm(x) + q.e_i$ is the value of the consumer's initial endowment, and $\theta_i[\int_L (P(x, b_y(x)) - P(x, 0))dm(x) + q.y]$ is the total profit share of individual i in the one firm of the economy. If b is a fixed vector, the problem can be solved exactly as in Berliant and ten Raa [7]. However, consumers generally are able to choose b freely (subject to the budget constraint), so it is desirable that b be a choice variable for them in the maximization problem. As far as the consumer is concerned, the maximization problem can be rewritten as follows:

$$\max_{(b,B,d)\in X} u_i(\int_B h_i(x, b(x))dm(x), d)$$

$$s.t. \int_B P(x, b(x))dm(x) + q.d \leq I_i.$$

The proof of the existence of a solution to the consumer's problem is similar to that for the producer's problem.

Theorem 2: *Let q_r be the r_{th} component of q. Under A2, if $q_r > 0$ for all $r = 1, ..., k$, there exists a solution to the consumer's problem.*

Proof: Since $(0, \phi, 0) \in X$ fulfills the budget constraint, there is a supremum value in the consumer's problem. Let $\{(b_t, B_t, d_t)\}_{t=1}^{\infty} \subseteq S_1 \times \mathcal{B} \times R_+^k$ be a maximizing sequence for the utility function tending to the supremum, and $\int_{B_t} P(x, b_t(x))m(x) + q.d_t \leq I_i$

for all t. The projection of the budget set onto its third component is compact in R^k since $q_j > 0$ for all $j = 1$ to k. Therefore, there exists a subsequence $\{d_{t_l}\}_{l=1}^{\infty}$ of $\{d_t\}_{t=1}^{\infty}$ converging to d^* in R^k. We pass to this subsequence, and denote it by $\{d_t\}_{t=1}^{\infty}$. Since d_t is non-negative in each component for all t, it follows that d^* is non-negative in each component.

By Ascoli's theorem, there exists a subsequence $\{b_{t_l}\}_{l=1}^{\infty}$ of $\{b_t\}_{t=1}^{\infty}$, which converges uniformly to $b^* \in S_1$. We pass to this subsequence, and we denote it by $\{b_t\}_{t=1}^{\infty}$. Hence, by the continuity of $P(., b(.))$ and $h_i(., b(.))$, and Lebesgue's dominated convergence theoorem, as in the proof of Theorem 1, it is straightforward to show that

$$\lim_{t\to\infty} \int_L P(x, b_t(x))dm(x) = \int_L P(x, b^*(x))dm(x)$$

and

$$\lim_{t\to\infty} \int_L h_i(x, b_t(x))dm(x) = \int_L h_i(x, b^*(x))dm(x).$$

Next, we show that there exists a subsequence $\{B_{t_l}\}_{l=1}^{\infty}$ in $\{B_t\}_{t=1}^{\infty}$ which converges to an appropriate limit $B^* \in \mathcal{B}$. We need convergence results for the following sequences:

$$\int_L 1_{B_t}(x)P(x, b^*(x))dm(x)$$
$$\int_L 1_{B_t}(x)P(x, 0)dm(x)$$
$$\int_L 1_{B_t}(x)\|b^*(x)\|_k dm(x)$$
$$\int_L 1_{B_t}(x)dm(x).$$

By Lyapunov's theorem, the image of $\mathcal{B}$ under the vector measure

$$\begin{pmatrix} \int P(x, b^*(x))dm(x) \\ \int P(x, 0)dm(x) \\ \int \|b^*(x)\|_k dm(x) \\ \int 1dm(x) \end{pmatrix}$$

is closed. Since P, b^*, and 1 are continuous functions defined on compact set L, their image is bounded. Thus the image of $\mathcal{B}$ under this vector measure is compact and there is a subsequence $\{B_{t_l}\}_{l=1}^{\infty}$ in $\{B_t\}_{t=1}^{\infty}$, converging to some $B^* \in \mathcal{B}$, that is

$$\lim_{l\to\infty}\int_{B_{t_l}} P(x, b^*(x))dm(x) = \int_{B^*} P(x, b^*(x))dm(x)$$
$$\lim_{l\to\infty}\int_{B_{t_l}} P(x, 0)dm(x) = \int_{B^*} P(x, 0)dm(x)$$
$$\lim_{l\to\infty}\int_{B_{t_l}} \|b^*(x)\|_k dm(x) = \int_{B^*} \|b^*(x)\|_k dm(x)$$
$$\lim_{l\to\infty}\int_{B_{t_l}} 1dm(x) = \int_{B^*} 1dm(x).$$

We pass to this subsequence, and we denote it by $\{B_t\}_{t=1}^{\infty}$. Next, we show that $b^*(x) = 0, \forall x \in B^{*c}$. Given $\epsilon > 0$, there exists T such that for every $t > T$, $\|b^*(x) - b_t(x)\|_k < \frac{\epsilon}{m(L)}$ for every $x \in L$ as $b_t \to b^*$ uniformly. Since $b_t(x) = 0, \forall x \in B_t^c$,

$$\|b^*(x)\|_k < \frac{\epsilon}{m(L)}, \forall t > T.$$

Hence $\forall t > T$,

$$\int_{B_t^c} \|b^*(x)\|_k dm(x) < \frac{\epsilon m(B_t^c)}{m(L)} \leq \epsilon.$$

Thus $\forall t > T$,

$$\int_L \|b^*(x)\|_k dm(x) - \int_{B_t} \|b^*(x)\|_k dm(x) < \epsilon,$$

and

$$\int_L \|b^*(x)\|_k dm(x) - \int_{B^*} \|b^*(x)\|_k dm(x) \leq \epsilon,$$

or

$$\int_{B^{*c}} \|b^*(x)\|_k dm(x) \leq \epsilon.$$

Since this is true for every $\epsilon > 0$, $\int_{B^{*c}} \|b^*(x)\|_k dm(x) = 0$. Thus $b^*(x) = 0, \forall x \in B^{*c}$.

Now consider the following equation:

$$\int_L 1_{B_t}(x)P(x,b_t(x))dm(x) - \int_L 1_{B^*}(x)P(x,b^*(x))dm(x)$$

$$= \int_L [1_{B_t}(x) - 1_{B^*}(x)]P(x,b^*(x))dm(x) + \int_L 1_{B_t}(x)[P(x,b_t(x)) - P(x,b^*(x))]dm(x).$$

Since $b_t \to b^*$ uniformly, given $\epsilon > 0$, there exists a $T > 0$ such that for all $t > T$, and all $x \in L$, $\|b^*(x) - b_t(x)\|_k < \epsilon$. Hence as $P \in S_2$, for $t > T$,

$$|1_{B_t}(x)[P(x,b_t(x)) - P(x,b^*(x))]| \leq c_2.1_{B_t}(x)\|b_t(x) - b^*(x)\|_k \leq c_2.\epsilon.1_{B_t}(x)$$

for almost all $x \in L$. For every $\delta > 0$, there exists a $T'' > 0$ such that for $t > \max(T, T'')$

$$\begin{aligned}
&|\int_L 1_{B_t}(x)[P(x,b_t(x)) - P(x,b^*(x))]dm(x)| \\
\leq &\int_L |1_{B_t}(x)[P(x,b_t(x)) - P(x,b^*(x))]|dm(x) \\
\leq &c_2 \int_L 1_{B_t}(x)\|[b_t(x) - b^*(x)]\|_k dm(x) \\
\leq &c_2.\epsilon. \int_L 1_{B_t}(x)dm(x) \\
\leq &c_2.\epsilon[\int_L 1_{B^*}(x)dm(x) + \delta].
\end{aligned}$$

Since this is true for every $\epsilon > 0$,

$$\lim_{t\to\infty} \int_L 1_{B_t}(x)[P(x,b_t(x)) - P(x,b^*(x))]dm(x) = 0.$$

Also, since $P \in L^1$,

$$\lim_{t\to\infty} \int_L [1_{B_t}(x) - 1_{B^*}(x)]P(x,b^*(x))dm(x) = 0,$$

and we get

$$\lim_{t\to\infty} \int_{B_t} P(x,b_t(x))dm(x) = \int_{B^*} P(x,b^*(x))dm(x).$$

Similarly, consider the following equation:

$$\int_L 1_{B_t}(x)h_i(x,b_t(x))dm(x) - \int_L 1_{B^*}(x)h_i(x,b^*(x))dm(x)$$

$$= \int_L [1_{B_t}(x) - 1_{B^*}(x)]h_i(x,b^*(x))dm(x) + \int_L 1_{B_t}(x)[h_i(x,b_t(x)) - h_i(x,b^*(x))]dm(x).$$

By Goffman [15] Theorem 7.4, and the fact that $P \in S_2$ and $\{b(x)|x \in L, b \in S_1\}$ is closed and bounded, h_i is uniformly continuous in its second argument. Thus, given $\epsilon > 0$, there exists a $T > 0$ such that for all $t > T$, and all $x \in L, |h_i(x,b^*(x)) - h_i(x,b_t(x))| < \epsilon$. Hence

$$|1_{B_t}(x)[h_i(x,b_t(x)) - h_i(x,b^*(x))]| \leq \epsilon.1_{B_t}(x)$$

for almost all $x \in L$. Since $m(B_t) \to m(B^*)$, for every $\delta > 0$, there exists a $T'' > 0$ such that for $t > \max(T, T'')$

$$\begin{aligned} &|\int_L 1_{B_t}(x)[h_i(x,b_t(x)) - h_i(x,b^*(x))]dm(x)| \\ \leq &\int_L |1_{B_t}(x)[h_i(x,b_t(x)) - h_i(x,b^*(x))]|dm(x) \\ \leq &\epsilon. \int_L 1_{B_t}(x)dm(x) \\ \leq &\epsilon[\int_L 1_{B^*}(x)dm(x) + \delta]. \end{aligned}$$

Since this is true for every $\epsilon > 0$,

$$\lim_{t\to\infty} \int_L 1_{B_t}(x)[h_i(x,b_t(x)) - h_i(x,b^*(x))]dm(x) = 0.$$

Since $h_i \in L^1$,

$$\lim_{t\to\infty} \int_L [1_{B_t}(x) - 1_{B^*}(x)]h_i(x,b^*(x))dm(x) = 0,$$

and we get

$$\lim_{t\to\infty} \int_{B_t} h_i(x,b_t(x))dm(x) = \int_{B^*} h_i(x,b^*(x))dm(x).$$

Given $\int_{B_t} P(x, b_t(x))dm(x) + q.d_t \leq I_i$ for all t, we have $\int_{B^*} P(x, b^*(x))dm(x) + q.d^* \leq I_i$. By the continuity of u_i in its $k+1$ arguments,

$$\lim_{t\to\infty} u_i(\int_{B_t} h_i(x, b_t(x))dm(x), d_t) = u_i(\int_{B^*} h_i(x, b^*(x))dm(x), d^*)$$

and this completes the proof.///

V. The Existence of an Equilibrium with Production

In this section we will prove the main theorem of the paper. The proof uses the fixed point theorem of Ky Fan [14] to obtain the existence of a competitive equilibrium. We define a competitive equilibrium for the production economy with land as follows.

Definition 7: *$((b_y, A, y)$, (b_1, B_1, d_1), ..., $(b_N, B_N, d_N))$ with $(b_y, A, y) \in Y$ and $(b_i, B_i, d_i) \in X$ for each i, is a feasible allocation if*

(a) $L = \cup_{i=1}^N B_i$ and $B_i \cap B_j = \phi$ a.s., $i \neq j$,

(b) $\sum_{i=1}^N d_i = \sum_{i=1}^N e_i + \int_A b_y(x)dm(x)$,

(c) $\sum_{i=1}^N b_i = b_y$,

(d) $b_i = 0$ *on* B_i^c *for all* i.

Given endowments $\{(0, E_i, e_i)\}_{i=1}^N$, the tuple $((b_y, A, y), (P, q)$, (b_1, B_1, d_1) ,..., $(b_N, B_N, d_N))$ with $(b_y, A, y) \in Y$, $(P, q) \in PS$, and $(b_i, B_i, d_i) \in X$ for each i, is a competitive equilibrium if $((b_y, A, y), (b_1, B_1, d_1), ..., (b_N, B_N, d_N))$ is a feasible allocation such that

(e) for each $i, (b_i, B_i, d_i)$ maximizes (3) subject to (4)

(f) (b_y, A, y) maximizes (2) over Y.

Since linear fractions of land parcels are not well-defined and linear combinations of land parcels are not necessarily in the σ-algebra of land parcels $\mathcal{B}$, the production and consumption sets are, therefore, not convex in any natural sense. Thus, we define land allocations and extend the utility function U_i for consumer i to the set of non-negative essentially bounded functions on land such that demand for fractions of parcels is allowed, and then reformulate the definitions of the consumption and production sets to be compatible with the usual notion of convexity.

Assumption A3: $(0, \phi, 0) \in Y$.

Definition 8:

(a) Let Ω' denote the subset of $S_1 \times L^\infty \times R^k$ where L^∞ is endowed with the weak topology induced by taking L^∞ as the dual of L^1 such that $(b, g, d) \in \Omega'$ if and only if $g \in G = \{s \in L^\infty(L) \mid 0 \leq s \leq 1, a.s.\}$.*

(b) Let $\bar{d}$ be the vector of highest possible quantities of mobile goods available to the economy and let D be the maximum of all the components in $\bar{d}$. The corresponding consumption set $X' \subseteq \Omega'$ is defined by $X' = \{(b, g, d) \in \Omega' \mid$ All components of d satisfy $D \geq d_j \geq 0$ for $j = 1, ..., k\}$. Let

$$U_i'(b, g, d) = u_i(\int_L [h_i'(x, b(x)) + g(x)h_i(x, 0)]dm(x), d)$$

where U_i' is defined on $S_1 \times L_+^\infty \times R_+^k$ and $h_i'(x, b(x)) = h_i(x, b(x)) - h_i(x, 0)$ for $x \in L$.

(c) The corresponding production set $Y' \subseteq \Omega'$ is defined by $Y' = \{(b, 1_L, y) \in \Omega' \mid (b, L, y) \in Y\}$.

Remark: *$\bar{d} < \infty$ because endowements of mobile goods are finite and if $(b, A, y) \in Y, b \in S_1$ and $\int_A b(x)dm(x)$ is bounded over b and A. It is clear that some $(b, g, d) \in X'$ can result in negative utility, but when $g = 1$, the utility is always positive. The modification of the definition of Y literally makes no change in the nature of the production, but will provide us with the desired property of convexity.*

We make the following assumption on utilities for the proof of the main theorem.

Assumption A4: *For all i, u_i is strictly monotone*[6] *and quasi-concave in its $k+1$ arguments, and for $\lambda, \lambda' \in R_+, 0 \leq \lambda + \lambda' \leq 1, b, b' \in S_1$ and $x \in L, h_i$ satisfies the following equality:*

$$\lambda(h_i(x, b(x)) - h_i(x, 0)) + \lambda'(h_i(x, b'(x)) - h_i(x, 0)) = h_i(x, (\lambda b + \lambda' b')(x)) - h_i(x, 0).$$

The property of the utility density h_i in A4 will be used in the proof of Lemmas 7 and 9. In accordance with the change of definitions of consumption and production sets, results regarding compactness of X' and Y' are proved below.

Lemma 1: *X' and Y' are compact in the product topology on $S_1 \times L^\infty \times R^k$.*

Proof: See appendix.

In our framework, a price P for land is interpreted in the following way. Given $b \in S_1$, P is a function which assigns to each $x \in L$ a non-negative real number $P(x, b(x))$. Note that even if there were a finite dimensional commodity space (i.e., land consisted of some finite number of plots), by changing b, it could appear that there were an infinite number of commodities in the model.

The existence proof takes into account the effect of production on $P(., b(.))$. We can be sure from theorems 1 and 2 that given $x \in L$, demand and supply are well-defined when a price system is chosen. Due to the non-linearity of the price system P in b, more attention than usual has to be paid to the behavior of P in order to get linearity of the budget set correspondence and the demand and supply correspondences (to be defined); we restrict the price systems to the space PS' defined below, where the price space for land is enlarged to price fractions of land parcels. If we can show an equilibrium exists with this restricted price set, then we need not worry about other prices.

Definition 9:

(a) $S_3 \subseteq C^0(L)$ is the set of Lipschitz functions mapping L to R_+ with constant $c_3 = \frac{c_2}{2} > 0$ where R_+ is the set of non-negative real numbers. $\rho \in S_3$ if and only if $\rho = 0$ on the boundary of L and $|\rho(x) - \rho(y)| \leq c_3 \|x - y\|_n$ for $x, y \in L$. S_3 is endowed with the uniform topology induced by the sup metric $sp(\rho_1, \rho_2) = \sup_{x \in L} |\rho_1(x) - \rho_2(x)|$.

(b) $S_4 \subseteq C^0(L \times R^k)$ is the set of Lipschitz functions mapping $L \times R^k$ to R_+ with constant $c_3 > 0$. Thus, $\rho' \in S_4$ if and only if $\rho' = 0$ on the boundary of L and $| \rho'(x, r_1) - \rho'(y, r_2) | \leq c_3(\|(x, r_1) - (y, r_2)\|_{n+k})$ for $x, y \in L$ and $r_1, r_2 \in R^k$. S_4 is endowed with the uniform topology.

Definition 10: *$PS' = \{(\rho, \rho', q) \in S_3 \times S_4 \times R^k_+ \mid \rho'$ is linear in $b \in S_1$, i.e., for $\lambda, \lambda' \in R, b, b' \in S_1$ and $x \in L$, $\lambda\rho'(x, b(x)) + \lambda'\rho'(x, b'(x)) = \rho'(x, (\lambda b + \lambda' b')(x))$, and $\forall b$,*

$$\int_L [\rho'(x, b(x)) + \rho(x)]dm(x) + \sum_{j=1}^{k} q_j = 1.\}$$

This is a rather mild restriction on land prices, which implies that a linear combination of the land price densities with input-output densities b and b' equals the land price density with linear combinations of the corresponding input-output densities. This notion of linearity is defined for each x in L and includes price systems such as $\rho'(x, b(x)) = C(x)b(x)$ for $x \in L$ where $C \in L^1(L)$ is continuous such that $\rho' \in S_4$. We price the land allocation and input-output density separately to avoid a "double linearity" problem, i.e., for $(b, g, d) \in \Omega'$, the price of land parcel (b, g) is

$\int_L[\rho'(x,b(x))+\rho(x)g(x)]dm(x)$. Notice that if $g(x)=1$ for $x \in L$, there is a $P \in S_2$ such that $P(x,b(x)) = \rho'(x,b(x)) + \rho(x)g(x)$. With price systems in PS', we will be able to show that the convexity assumption of the Ky Fan theorem is satisfied.

Lemma 2: *PS' is compact in the product topology on $S_3 \times S_4 \times R^k_+$.*

Proof: See appendix.

As consumers are allowed to buy fractions of points and $(\rho,\rho',q) \in PS'$, the corresponding producer's and consumer's problems are

$$\max_{(b_y,1_L,y)\in Y'} \int_L \rho'(x,b(x))dm(x) + q.y \tag{2A}$$

and

$$\max_{(b,g,d)\in X'} u_i(\int_L [h_i'(x,b(x)) + g(x)h_i(x,0)]dm(x), d) \tag{3A}$$

$$s.t. \int_L [\rho'(x,b(x)) + \rho(x)g(x)]dm(x) + q.d \le I_i, \tag{4A}$$

respectively. The corresponding concept of competitive equilibrium, "mixed equilibrium", is presented in Definition 11. We formulate demand, supply, and excess demand correspondences in Definition 12.

Definition 11: *$((b_y,1_L,y),(b_1,g_1,d_1),...,(b_N,g_N,d_N))$ with $(b_y,1_L,y) \in Y'$ and $(b_i,g_i,d_i) \in X'$ for each i, is a mixed feasible allocation if*

(a)$\sum_{i=1}^N g_i = 1_L$,
(b)$\sum_{i=1}^N d_i = \sum_{i=1}^N e_i + \int_L b_y(x)dm(x)$,
(c)$\sum_{i=1}^N b_i = b_y$.

Given the endowment $\{(0,E_i,e_i)\}_{i=1}^N$, $((b_y,1_L,y),(\rho,\rho',q),(b_1,g_1,d_1),...,(b_N,g_N,d_N))$ with $(b_y,1_L,y) \in Y'$, $(\rho,\rho',q) \in PS'$, and $(b_i,g_i,d_i) \in X'$ for each i, is a mixed *equilibrium if* $((b_y, 1_L, y), (b_1, g_1, d_1), ..., (b_N, g_N, d_N))$ *with* $(b_y, 1_L, y) \in Y', (\rho, \rho', q) \in PS'$, *and* $(b_i, g_i, d_i) \in X'$ *for each* i, *is a mixed feasible allocation such that*

(d) for each $i, (b_i, g_i, d_i)$ *maximizes* U_i' *on budget set* $\chi_i(\rho, \rho', q, I_i)$*(defined below),*

(e) $(b_y, 1_L, y)$ *maximizes profit relative on* Y' *to* (ρ, ρ', q).

Definition 12:

(a) The budget set correspondence of consumer i, χ_i, *from* $PS' \times R_+$ *to* X' *is defined by* $\chi_i(\rho, \rho', q, I_i) = \{(b, g, d) \in X' \mid \int_L [\rho'(x, b(x)) + \rho(x)g(x)]dm(x) + q.d \leq I_i\}$ *where* I_i *is the total wealth of consumer* i *as defined in (4).*

(b) The demand correspondence ξ_i *for consumer* i *from* $PS' \times R_+$ *to* X' *is defined by* $\xi_i(\rho, \rho', q, I_i) = \{(b, g, d) \in X' \mid (b, g, d)$ *is a greatest element of* χ_i *under* $u_i\}$ *if* $I_i \neq 0$, *and* $\{(b, g, d) \in X' \mid (b, g, d)$ *is an element of* $\chi_i(\rho, \rho', q, 0)$, *at least as good as the endowment* $(0, 1_{E_i}, e_i)$ *under* $u_i\}$ *if* $I_i = 0$. *The aggregate demand correspondence is* $\xi(\rho, \rho', q, I) = \sum_{i=1}^{N} \xi_i(\rho, \rho', q, I_i)$ *where* $I = (I_1, ..., I_N)$.

(c) The supply correspondence, η, *from* PS' *to* Y' *is defined by* $\eta(\rho, \rho', q) = \{(b, 1_L, y) \in Y' \mid (b, 1_L, y)$ *maximizes the profit function (2A)* $\}$.

(d) The set of excess demands Z *is a subset of* $S_1 \times L^\infty \times R^k$ *and is defined as* $Z = X' - Y' + \{(0, 1_L, 0)\} - \{\sum_{i=1}^{N}(0, 1_{E_i}, e_i)\}$ *where* 1_{E_i} *is the land endowment of consumer* i. *The excess demand correspondence,* ζ, *from* PS' *to* $S_1 \times L^\infty \times R^k$ *is defined by* $\zeta(\rho, \rho', q) = \{z \in Z \mid z \in \xi(\rho, \rho', q, I) - \eta(\rho, \rho', q) + (0, 1_L, 0) - \sum_{i=1}^{N}(0, 1_{E_i}, e_i)\}$.

(e) Let $(b_i, g_i, d_i) \in \xi_i(\rho, \rho', q)$ *and* $(b_y, 1_L, y) \in \eta(\rho, \rho', q)$. *Define the value of the*

excess demand $z \in \zeta(\rho, \rho', q)$ as

$$V_z(\rho,\rho',q) = \{\sum_{i=1}^{N}\int_L [\rho'(x,b_i(x)) + g_i(x)\rho(x)]dm(x) + q.\sum_{i=1}^{N} d_i\}$$
$$-\{\int_L \rho'(x,b_y(x))dm(x) + \int_L \rho(x)dm(x) + q.[\sum_{i=1}^{N} e_i + \int_L b_y(x)dm(x)]\}$$

$$= \int_L [\rho'(x,(\sum_{i=1}^{N} b_i - b_y)(x)) + (\sum_{i=1}^{N} g_i - 1_L)(x)\rho(x)]dm(x) + q.(\sum_{i=1}^{N} d_i - \sum_{i=1}^{N} e_i - y)$$
$$= \int_L [\rho'(x,b_z(x)) + g_z(x)\rho(x)]dm(x) + q.d_z$$

where $b_z = \sum_{i=1}^{N} b_i - b_y$, $g_z = \sum_{i=1}^{N} g_i - 1_L$, and $d_z = \sum_{i=1}^{N} d_i - \sum_{i=1}^{N} e_i - y$. *The correspondence, τ, from $S_1 \times L^\infty \times R^k$ to PS' is defined by $\tau(z) = \{(\rho,\rho',q) \in PS' \mid (\rho,\rho',q)$ maximizes the value of excess demand $V_z(\rho,\rho',q)\}$.*

Definition 13:

(1) A correspondence f of a metric space S into a metric space T is said to be closed if the graph of f is a closed subset of $S \times T$ in the product topology.

(2) A correspondence f of a metric space S into a metric space T is said to be upper hemi-continuous (u.h.c.) at $x \in S$ if $f(x) \neq \phi$ and if for every neighborhood U of $f(x)$ there exists a neighborhood V of x such that $f(z) \subseteq U$ for every $z \in V$. f is u.h.c. if it is u.h.c. at every $x \in S$.

Lemma 3: *Under A2, $\chi_i(\rho,\rho',q,I_i)$ is a closed correspondence.*

Proof: See appendix.

Remarks: (i) By Lemma 3, for any $(\rho,\rho',q,I_i) \in PS' \times R_+$, $\chi_i(\rho,\rho',q,I_i)$ is closed. Since $\chi_i(\rho,\rho',q,I_i)$ is a closed subset of a compact set X', $\chi_i(\rho,\rho',q,I_i)$ is compact-valued. (ii) Since $(0,\phi,0) \in \chi_i$ for all $(\rho,\rho',q) \in PS'$, χ_i is nonempty. ξ_i is non-empty as χ_i is non-empty and compact-valued, and the utility function is continuous. (iii) η is non-empty as Y' is non-empty and compact, and the profit function is continuous. (iv) ζ is non-empty as the sum of non-empty sets. (v) τ is non-empty as PS' is non-empty and compact by Lemma 2. (vi) Z is compact as a sum of a finite number of compact sets.

Lemma 4: *Let $\{(\rho_t, \rho'_t, q_t, I_t)\}_{t=1}^{\infty}$ be a sequence in the set $PS' \times R_+$ converging to $(\rho^*, \rho'^*, q^*, I^*)$ and let $(b^*, g^*, d^*) \in \chi_i(\rho^*, \rho'^*, q^*, I^*)$. Under A1, if I^* is not the minimum wealth relative to (ρ^*, ρ'^*, q^*), there exists a sequence $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$ such that $(b_t, g_t, d_t) \in \chi_i(\rho_t, \rho'_t, q_t, I_t)$ for all t and $(b_t, g_t, d_t) \rightarrow (b^*, g^*, d^*)$ as $t \rightarrow \infty$.*

Proof: See appendix.

Lemma 5: *Under A2 and for $(\rho, \rho', q, I_i) \in PS' \times R_+$, $\xi_i(\rho, \rho', q, I_i)$ and $\eta(\rho, \rho', q)$ are upper hemi-continuous.*

Proof: See appendix.

Lemma 6: *$\tau(z)$ is upper hemi-continuous and compact-valued.*

Proof: See appendix.

An important feature of the consumption set and the production set is convexity, without which the fixed point theorem of Ky Fan [14] will not be applicable to our model. By convexity of Y', we mean that for any $(b, 1_L, y)$ and $(b', 1_L, y') \in Y', (\lambda b + (1-\lambda)b', 1_L, \lambda y + (1-\lambda)y') \in Y'$ where $0 \leq \lambda \leq 1$. That is, any linear combination of feasible technologies is still feasible.

With land allocations being in G as discussed in the beginning of this section, convexity of the consumption and production sets can be well-defined and it is straightforward to show that X' is convex as Ω' and the set of mobile goods allocations are convex. Next, we make the assumption that a subset of Yi is convex, and show that it implies that Y' is also convex.

Assumption A5: *$D = \{(b_y, A, y) \in Y \mid A = L\}$ is convex in b and y.*

Remark: Since D is convex in b and y, for (b, L, y), $(b', L, y') \in Y$, and $0 \leq \lambda \leq 1$, $(\lambda b + (1-\lambda)b', L, \lambda y + (1-\lambda)y') \in Y$. By (c) of Definition 8, $(b, 1_L, y), (b', 1_L, y')$ and $(\lambda b + (1-\lambda)b', 1_L, \lambda y + (1-\lambda)y')$ all belong to Y'. Hence, Y' is convex.

In view of the convexity and continuity assumptions, we can prove the following result.

Lemma 7: *Under $A2, A4$ and $A5$, $\xi_i(\rho, \rho', q, I_i)$ and $\eta(\rho, \rho', q)$ are convex-valued.*

Proof: See appendix.

The strategy of the proof of existence of an equilibrium is as follows. First, we check the assumptions of the theorem of Ky Fan [14] to find a mixed equilibrium for an economy with price vector $(\rho, \rho', q) \in PS'$ and then we can show, by using an extreme point argument of Berliant [5], that one mixed equilibrium is actually a competitive equilibrium.

Theorem 4: *If the economy E satisfies assumptions A1-A5, then there exists a mixed equilibrium for E.*

Proof: In order to obtain a mixed equilibrium, we use U' and then check the assumptions of the theorem of Fan [14]. The price space PS' is compact by Lemma 2. From the proof of Lemma 5, ξ_i and η are closed correspondences. Since X' and Y' are compact, ξ_i and η are compact-valued. By Lemma 5, they are also upper hemi-continuous. Using Klein and Thompson [18] Theorem 7.3.15, ξ and ζ are upper hemi-continuous and compact-valued. Since ξ_i and η are convex-valued, so are ξ and ζ.

By (iv) and (v) of the remarks after Lemma 3, both ζ and τ are non-empty. By Lemma 6, $\tau(z)$ is upper hemi-continuous and compact-valued. Since V_z is linear in (ρ, ρ', q), $\tau(z)$ is convex-valued. Now let ψ denote a correspondence $\psi((\rho, \rho', q), z) = \tau(z) \times \zeta(\rho, \rho', q)$. Using Klein and Thompson [18] Theorem 7.3.14, ψ is upper hemi-continuous. Moreover, ψ is non-empty and convex-valued as the product of two non-empty, convex-valued correspondences.

$PS' \times Z$ is a particular subset of a locally convex linear topological space, $PS' \times Z$ is non-empty, compact and convex, and $\psi((\rho, \rho', q), z)$ is a non-empty, upper hemi-continuous, and convex-valued correspondence mapping $PS' \times Z$ to $PS' \times Z$. We apply Ky Fan's fixed point theorem [14] and get a fixed point $((\rho^*, \rho'^*, q^*), z^*)$ for ψ. Thus,

$$((\rho^*, \rho'^*, q^*), z^*) \in \psi((\rho^*, \rho'^*, q^*), z^*) = \tau(z^*) \times \zeta(\rho^*, \rho'^*, q^*)$$

which is equivalent to $(\rho^*, \rho'^*, q^*) \in \tau(z^*)$ and $z^* \in \zeta(\rho^*, \rho'^*, q^*)$. Since $z^* = (b_z^*, g_z^*, d_z^*) \in \zeta(\rho^*, \rho'^*, q^*)$, there are $(b_i^*, g_i^*, d_i^*) \in \xi_i$ for each i, and $(b_y^*, 1_L, y^*) \in \eta$ such that

$$g_z^* = \sum_{i=1}^{N} g_i^* - 1_L,$$

$$d_z^* = \sum_{i=1}^{N} d_i^* - \sum_{i=1}^{N} e_i - \int_L b_y^*(x) dm(x),$$

and

$$b_z^* = \sum_{i=1}^{N} b_i^* - b_y^*.$$

Let I_i^* denote $\int_{E_i} \rho^*(x) dm(x) + q^*.e_i + \theta_i[\int_L \rho'^*(x, b_y^*(x)) dm(x) + q^*. \int_L b_y^*(x) dm(x)]$. By strict monotonicity of the utility function, we have, for each i,

$$\int_L [\rho'^*(x, b_i^*(x)) + g_i^*(x)\rho^*(x)] dm(x) + q^*.d_i^*$$

$$= \int_{E_i} \rho^*(x)dm(x) + q^*.e_i + \theta_i[\int_L \rho^{'*}(x, b_y^*(x))dm(x) + q^*. \int_L b_y^*(x)dm(x)]$$

Summing over i, we obtain,

$$\sum_{i=1}^{N} \int_L [\rho^{'*}(x, b_i^*(x)) + g_i^*(x)\rho^*(x)]dm(x) + q^*. \sum_{i=1}^{N} d_i^*$$

$$= \int_L \rho^*(x)dm(x) + q^*. \sum_{i=1}^{N} e_i + \int_L \rho^{'*}(x, b_y^*(x))dm(x) + q^*. \int_L b_y^*(x)dm(x)$$

$$= \int_L \rho^{'*}(x, b_y^*(x))dm(x) + \int_L \rho^*(x)dm(x) + q^*.[\sum_{i=1}^{N} e_i + \int_L b_y^*(x)dm(x)]$$

which implies that the value of excess demand V_{z^*} at $(\rho^*, \rho^{'*}, q^*)$ is 0, i.e.,

$$V_{z^*}(\rho^*, \rho^{'*}, q^*) = \int_L [\rho^{'*}(x, b_z^*(x)) + g_z^*(x)\rho^*(x)]dm(x) + q^*.d_z^* = 0.$$

First we show that $\forall i, I_i^* > 0$. If $I_i^* = 0$, since $e_i > 0$ in some component, q is zero in that component. By A4, there is excess demand for this commodity or $I_i^* = 0 \forall i$. $I_i^* = 0 \forall i$ is impossible by definition of PS'. Hence, there is excess demand for this commodity, say mobile good c. But then $(\rho^*, \rho^{'*}, q^*)$ does not maximize the value of excess demand. The price system where $\rho = \rho' = 0$, and $q_c = 1$ for mobile good c and $q_j = 0$ for $j \neq c$ gives excess demand a higher value which is positive, thus a contradiction. So $I_i^* > 0$. Since $(\rho^*, \rho^{'*}, q^*)$ maximizes the value of the excess demand $V_{z^*}(\rho, \rho', q)$, every component of the excess demand for mobile goods d_z^* is less than or equal to 0, as V_{z^*} is less than or equal to 0 at $\rho = \rho' = 0$, and $q_t = 1$ for some t and $q_j = 0$ for $j \neq t$. However, if the t_{th} component of d_z^* is strictly less than zero, it must be that $q_t^* = 0$, since q^* maximizes the value of excess demand on d_z^*. Strict monotonicity contradicts the assumption that the t_{th} component of d_z^* is strictly less than zero. So the excess demand for mobile goods, d_z^*, is zero.

Suppose $\rho'^*(x, b_z^*(x)) < 0$ on some set of positive measure in L. Consider a function $C(x)$ such that

$$C(x) = \begin{cases} 1, & \text{if } \rho'^*(x, b_z^*(x)) \geq 0; \\ -1, & \text{if } \rho'^*(x, b_z^*(x)) < 0. \end{cases}$$

Notice that $C.\rho'^* \in S_4$ and

$$\int_L C(x)\rho'^*(x, b_z^*(x))dm(x) > \int_L \rho'^*(x, b_z^*(x))dm(x),$$

which is a contradiction to the fact that (ρ^*, ρ'^*, q^*) maximizes V_{z^*}. Hence $\rho'^*(x, b_z^*(x)) \geq 0$. Now if $\rho'^*(x, b_z^*(x)) > 0$ on some set of positive measure in L, by setting $\rho = 0$, $V_{z^*}(\rho, \rho'^*, q^*) > 0$ which contradicts that (ρ^*, ρ'^*, q^*) maximizes V_{z^*}. Thus we have established that $\rho'^*(x, b_z^*(x)) = 0$ which implies that $\int_L g_z^*(x)\rho^*(x)dm(x) = 0$.

Next we show $g_z^* = 0$ a.s. Let $\Gamma = \{x \in L \mid g_z^*(x) > 0\}$ and suppose $m(\Gamma) > 0$. Let $\Gamma_\epsilon = \cup_{x\in\Gamma} B_\epsilon(x)$ where $\epsilon > 0$ and $B_\epsilon(x)$ is the ϵ ball around x in L. Choose an $\epsilon' > 0$ such that $m(\Gamma_{\epsilon'}/\Gamma) < \frac{1}{N}\int_\Gamma g_z^*(x)dm(x)$ (this is possible since m is a regular measure) and construct $\hat{\rho} \in S_3$ as follows.

$$\hat{\rho}(x) = \begin{cases} c_3.\min(\epsilon', \inf_{y\in(\Gamma_{\epsilon'})^c} \|x - y\|_n), & \text{if } x \in \Gamma_{\epsilon'}; \\ 0, & \text{otherwise.} \end{cases}$$

Then, since $|g_z^*(x)| \leq N$ a.s.,

$$\begin{aligned} \int_L g_z^*(x)\hat{\rho}(x)dm(x) &= \int_{\Gamma_{\epsilon'}} g_z^*(x)\hat{\rho}(x)dm(x) \\ &= \int_\Gamma g_z^*(x)\hat{\rho}(x)dm(x) + \int_{\Gamma_{\epsilon'}/\Gamma} g_z^*(x)\hat{\rho}(x)dm(x) \\ &\geq c_3.\epsilon' \int_\Gamma g_z^*(x)dm(x) - c_3.\epsilon'.N.m(\Gamma_{\epsilon'}/\Gamma) \\ &= c_3.\epsilon'(\int_\Gamma g_z^*(x)dm(x) - N.m(\Gamma_{\epsilon'}/\Gamma)) \\ &> 0, \end{aligned}$$

which is a contradiction. Thus, $m(\Gamma) = 0$. Suppose $g_z^*(x) < 0$ on some set of positive measure in L. Since ρ^* is nonnegative a.s. and $g_z^* > 0$ only on a set of measure zero, $\int_L g_z^*(x).\rho^*(x)dm(x) < 0$ which contradicts that $\int_L g_z^*(x).\rho^*(x)dm(x) = 0$. Hence $g_z^* = 0$ a.s.

Finally we show that $b_z^* = 0$ a.s. Let $\Gamma' = \{x \in L \mid b_z^*(x) \neq 0\}$ and suppose $m(\Gamma') > 0$. Let $x' \in \Gamma'$. Therefore, there is an $i \in \{1, ..., k\}$ such that the i_{th} component of $b_z^*(x')$, call it $b_{z,i}^*(x')$, is either positive or negative. Suppose $b_{z,i}^*(x') > 0$. Then there exists an $\hat{\epsilon} > 0$ such that $b_{z,i}^*(x') > \hat{\epsilon} > 0$. Construct $\rho' \in S_4$ as follows.

$$\rho'(x, b_z^*(x)) = \begin{cases} \frac{c_3}{c_1} b_{z,i}^*(x), & \text{if } x \in B_{\bar{\epsilon}}(x'); \\ 0, & \text{otherwise,} \end{cases}$$

where $\hat{\epsilon} > 2c_1\bar{\epsilon} > 0$. It is clear that for any $x, y \in B_{\bar{\epsilon}}(x')$,

$$|b_{z,i}^*(x) - b_{z,i}^*(y)| \leq c_1\|x - y\|_n \leq 2c_1.\bar{\epsilon} < \hat{\epsilon},$$

which implies that $b_{z,i}^*$ does not change sign in $B_{\bar{\epsilon}}(x')$, and one has

$$\int_L \rho'(x, b_z^*(x))dm(x) = \int_{B_{\bar{\epsilon}}(x')} \rho'(x, b_z^*(x))dm(x) = \frac{c_3}{c_1}\int_{B_{\bar{\epsilon}}(x')} b_{z,i}^*(x)dm(x) > 0$$

which contradicts that $\int_L {\rho'}^*(x, b_z^*(x))dm(x) = 0$ is the maximum value. Similarly, if $b_{z,i}^*(x') < 0$, change the coefficient attached to $b_{z,i}^*$ in ρ' to $-\frac{c_3}{c_1}$, and this also leads to contradiction. Thus, $b_z^* = 0$ a.s. Since $g_z^* = 0$ a.s. and $b_z^* = 0$ a.s., and $d_z^* = 0$, one has

$$\begin{aligned} \sum_{i=1}^{N} g_i^* &= 1_L, a.s. \\ \sum_{i=1}^{N} d_i^* &= \sum_{i=1}^{N} e_i + y^*, \qquad (5) \\ \sum_{i=1}^{N} b_i^* &= b_y^*, a.s. \end{aligned}$$

By (5) and that (b_i^*, g_i^*, d_i^*) maximizes utility for each i subject to the budget constraint, $[(b_y^*, 1_L, y^*),\ (\rho^*, {\rho'}^*, q^*),\ (b_1^*, g_1^*, d_1^*), ...,\ (b_N^*, g_N^*, d_N^*)]$ is a mixed equilibrium by definition./ / /;

Remark: We have shown $I_i^* > 0 \forall i$ in a mixed equilibrium. This is used in many of the lemmas below.

In the next lemma, we show that if any set of mixed equilibrium allocations has an xtreme point, then the land allocation is an indicator function for all consumers, i.e., no one is sharing any point of his land with any one else.

Lemma 8: *Let $ME(\rho, \rho', q; d, b_y) \subseteq (S_1)^N \times (L_+^\infty(L))^N$ be the set of consumers' consumption of land in a mixed equilibrium associated with $(\rho, \rho', q) \in PS'$, mobile goods allocation d and production input-output density b_y. If $\{(b_i, g_i)\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$ is an extreme point of $ME(\rho, \rho', q; d)$, then g_i is an indicator function for all $i, 1 \leq i \leq N$.*

Proof : See appendix.

Lemma 9: *$ME(\rho, \rho', q; d, b_y)$ is convex and compact in the product topology on $(S_1)^N \times (L^\infty(L))^N$.*

Proof: See appendix.

In the proof of the existence of a mixed equilibrium, no restriction was imposed on g and b for each consumer to rule out a phenomenon not observed in reality, namely, consumers may have non-zero input-output density on some set of land of positive measure which they do not own. By the following assumption and Lemma 8, we can show that if $\{(b_i, g_i)\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$ is an extreme point of $ME(\rho, \rho', q; d, b_y)$, then $g_i(x) \neq 0$ a.s. whenever $b_i'(x) \neq 0$ a.s. for all i. That is, no consumer has non-zero input-output densities on the land he does not own.

Assumption A6: *Let X_i be the consumption set for consumer i and let $d_{i,t}$ and $b_{i,t}$ be the t_{th} components of d_i and b_i, respectively, where $t = 1, ..., k$. Assume that u_i is twice differentiable. Define*

$$U_i^0(b_i, B_i, d_i) = \frac{\partial u_i(a, d_i)}{\partial a}|_{a=\int_{B_i} h_i(x, b_i(x))dm(x)}$$

and

$$U_i^t(b_i, B_i, \bar{d}_i) = \frac{\partial u_i(a, d_i)}{\partial d_{i,t}}|_{a=\int_{B_i} h_i(x, b_i(x))dm(x), d_i=\bar{d}_i}.$$

Postulate:

(1) $\forall(b_i, B_i, d_i) \in X_i, U_i^0 > 0$ *for every* i, *and* $\forall(b_j, B_j, d_j) \in X_j, \forall s > 0$,

$$m(\{x \in L | U_i^0(b_i, B_i, d_i)h_i(x, 0) = sU_j^0(b_j, B_j, d_j)h_j(x, 0)\}) = 0.$$

(2) There is a mobile good l *such that for* $(b_i, B_i, d_i), (b'_i, B'_i, d'_i) \in X_i$, and $d'_{i,l} \neq 0$, $\forall i = 1, ..., N$, $j = 0, ..., k$, $\lim_{d_{i,l} \to 0} U_i^l(b_i, B_i, d_i) > U_i^j(b'_i, B'_i, d'_i)$.

(3) Let $B \in \mathcal{B}$ *with* $m(B) \neq 0$. h_i *is differentiable with respect to* b_i *for all* i. *For all* $(b_i, B_i, d_i) \in X_i, (b_j, B_j, d_j) \in X_j$, *if there is a mobile good* l *such that*

$$\frac{U_i^0(b_i, B_i, d_i)h_i(x, 0)}{U_i^l(b_i, B_i, d_i)} > \frac{U_j^0(b_j, B_j, d_j)h_j(x, 0)}{U_j^l(b_j, B_j, d_j)}$$

where $x \in B$ *and* $i \neq j$, *then*

$$\frac{U_i^0(b_i, B_i, d_i)|\frac{\partial h_i(x, b_i(x))}{\partial b_{i,l}(x)}|}{U_i^l(b_i, B_i, d_i)} > \frac{U_j^0(b_j, B_j, d_j)|\frac{\partial h_j(x, b_j(x))}{\partial b_{j,l}(x)}|}{U_j^l(b_j, B_j, d_j)}.$$

Part 1 of assumption A6 means that no two consumers have utility densities over land proportional to each other. Part 2 is similar to an Inada condition that assumes consumption of mobile good l is essential to every consumer. The last part states that if one consumer has higher utility density ratio for some land than any other consumer, then his utility density ratio for input-output density is also higher than anyone else's. One example satisfying this assumption is $U_i(b_i, B_i, d_i) = u_i(\int_{B_i} h_i(x, b_i(x))dm(x), d_i)$ where $h_i(x, b_i(x)) = T_i(x)\sum_{t=1}^{k}(-b_{i,t}(x)) + T_i(x)$, $T_i \in L^1$ and $T_i > 0$ a.s..

<u>Lemma 10</u>: *Under A6, if* $\{(b_i, g_i)\}_{i=1}^{N} \in ME(\rho, \rho', q; d, b_y)$ *is an extreme point of* $ME(\rho, \rho', q; d, b_y)$*, then for all* i*,* $g_i(x) > 0$ a.s. *whenever* $b_i(x) \neq 0$.

Proof: See appendix.

<u>Theorem 5</u>: *If the economy E satisfies assumptions A1-A6, then there exists a competitive equilibrium for E.*

Proof: By Theorem 4, there exists a mixed equilibrium for economy E. Let $ME(\rho, \rho', q; d, b_y) \subseteq (S_1)^N \times (L^\infty(L))^N$ be the set of consumers' land allocations of the mixed equilibrium associated with $(\rho, \rho', q) \in PS'$ and mobile goods $d \in (R_+^k)^N$. Then by Lemma 9, $ME(\rho, \rho', q; d, b_y)$ is convex and compact in the topology on $(S_1)^N \times (L^\infty(L))^N$. By the Krein-Milman Theorem (Rudin [20] p.70), the consumers' land allocation in the mixed equilibrium, $ME(\rho, \rho', q; d, b_y)$, is the closed convex hull of its extreme points. Since $ME(\rho, \rho', q; d, b_y) \neq \phi, ME(\rho, \rho', q; d, b_y)$ has an extreme point. By Lemmas 8 and 10, an extreme point of $ME(\rho, \rho', q; d, b_y)$ can be represented by a vector of input-output densities and indicator functions of land parcels, call it $\{(b_1, 1_{B_1}), ..., (b_N, 1_{B_N})\}$ where $B_i \in \mathcal{B}$ and $1_{B_i}(x) \neq 0$ a.s. whenever $b_i(x) \neq 0$ a.s. for

$x \in L$ and each i. Thus it is straightforward to show

$$\int_L [h_i'(x, b_i(x)) + g_i(x)h_i(x,0)]dm(x) = \int_L [h'(x, b_i(x)) + 1_{B_i}(x)h_i(x,0)]dm(x)$$
$$= \int_{B_i} h_i(x, b_i(x))dm(x)$$

and there is a $(P,q) \in PS$ such that

$$\int_L [\rho'(x, b_i(x)) + g_i(x)\rho(x)]dm(x) = \int_L [\rho'(x, b_i(x)) + 1_{B_i}(x)\rho(x)]dm(x)$$
$$= \int_{B_i} P(x, b_i(x))dm(x).$$

We conclude that (b_i, B_i, d_i) also solves the consumer's problem (3) subject to (4) for each i.

Let $A = \{x \in L \mid b_y(x) \neq 0\}$. It is obvious that $y = \int_L b_y(x)dm(x) = \int_A b_y(x)dm(x)$, hence $(b_y, A, y) \in Y$ and makes the same profit as $(b_y, 1_L, y)$ relative to $(P,q) \in PS$, and therefore it solves the producer's problem. Given (a) of (5), one has $\sum_{i=1}^N 1_{B_i}(x) = 1_L(x) = 1$ for every $x \in L$, and thus $B_i \cap B_j = \phi$ for $i \neq j, 1 \leq i,j \leq N$. Also, $\cup_{i=1}^N B_i = L$. Thus, $[(b_y, A, y), (b_1, B_1\, d_1), ..., (b_N, B_N, d_N)]$ is a feasible allocation as (a), (b), (c), and (d) of Definition 7 are satisfied. By the definition of a competitive equilibrium, $((b_y, A, y), (P,q), (b_1, B_1, d_1), ..., (b_N, B_N, d_N))$ is a competitive equilibrium. This completes the proof.///

Definition 14: *A feasible allocation* $[(b_y, A, y), (b_1, B_1, d_1), ..., (b_N, B_N, d_N)]$ *is Pareto optimal if for any other feasible allocation* $[(b_y', A', y'), (b_1', B_1', d_1'), ..., (b_N', B_N', d_N')]$, *it is not true that* $U_i(b_i', B_i', d_i') \geq U_i(b_i, B_i, d_i)$ *for all* $i = 1, ..., N$, *and* $U_j(b_j', B_j', d_j') > U_j(b_j, B_j, d_j)$ *for some* j.

Definition 15: *A utility* U *over* X *is called locally non-satiated if for any* $(b, B, d) \in X$, *and* $\epsilon > 0$, *there is* $(b', B', d') \in X$ *with* $\|d - d'\|_k < \epsilon$, $\int_L |1_B(x) - 1_{B'}(x)|dm(x) < \epsilon$, *and* $\|b(x) - b'(x)\|_k < \epsilon$ *for all* $x \in L$, *such that* $U(b', B', d') > U(b, B, d)$.

We prove the first welfare theorem in theorem 6.

Theorem 6: *Under A1, if each consumer has a utility satisfying A2 and that is locally non-satiated, then any equilibrium allocation is Pareto optimal.*

Proof: Given endowments $\{(0, E_i, e_i)\}_{i=1}^N$, let $((b_y, A, y), (P, q), (b_1, B_1, d_1), \ldots, (b_N, B_N, d_N))$ with $(b_y, A, y) \in Y$, $(P, q) \in PS$, and $(b_i, B_i, d_i) \in X$ for each i, be a competitive equilibrium. Let $[(b'_y, A', y'), (b'_1, B'_1, d'_1), \ldots, (b'_N, B'_N, d'_N)]$ be a feasible allocation Pareto-dominating the equilibrium allocation.

If $U_i(b'_i, B'_i, d'_i) > U_i(b_i, B_i, d_i)$ for some i, by the definition of an equilibrium,

$$\int_{B'_i} P(x, b'_i(x))dm(x) + q.d'_i > \int_{B_i} P(x, b_i(x))dm(x) + q.d_i.$$

Suppose $U_i(b'_i, B'_i, d'_i) = U_i(b_i, B_i, d_i)$ for some i. Since consumer i is locally non-satiated, there exists a sequence $\{(b_i^t, B_i^t, d_i^t)\}_{t=1}^{\infty} \in X$ such that $\forall \epsilon > 0$ there exists a $T > 0$ such that $\forall t \geq T$, $\|d_i^t - d'_i\|_k < \epsilon$, $\int_L |1_{B_i^t}(x) - 1_{B'_i}(x)| dm(x) < \epsilon$, and $\|b_i^t(x) - b'_i(x)\|_k < \epsilon$ for all $x \in L$, and $U_i(b_i^t, B_i^t, d_i^t) > U_i(b'_i, B'_i, d'_i)$. By the definition of an equilibrium,

$$\int_{B_i^t} P(x, b_i^t(x))dm(x) + q.d_i^t > \int_{B_i} P(x, b_i(x))dm(x) + q.d_i. \tag{6}$$

Taking the limit of (6) as $t \to \infty$ and using the Lebesgue's dominated convergence theorem, we have

$$\int_{B'_i} P(x, b'_i(x))dm(x) + q.d'_i \geq \int_{B_i} P(x, b_i(x))dm(x) + q.d_i.$$

By summing the budget constraints over i, one has

$$\sum_{i=1}^{N}\int_{B_i'} P(x,b_i'(x))dm(x)+q.\sum_{i=1}^{N}d_i'$$
$$>\sum_{i=1}^{N}\int_{B_i} P(x,b_i(x))dm(x)+q.\sum_{i=1}^{N}d_i.$$
$$=\int_A [P(x,b_y(x))-P(x,0)]dm(x)+q.y+\sum_{i=1}^{N}\int_{E_i} P(x,0)dm(x)+q^*.\sum_{i=1}^{N}e_i.$$

And thus,

$$\sum_{i=1}^{N}\int_{B_i'}[P(x,b_i'(x))-P(x,0)]dm(x)+q.\sum_{i=1}^{N}[d_i'-e_i]$$
$$>\int_A [P(x,b_y(x))-P(x,0)]dm(x)+q.y,$$

which contradicts that (b_y, A, y) maximizes profit at (P,q). This completes the proof.///

VI. Conclusions

Questions arise as to the number of the producers in the model. The results will still be valid in models with more than one producer as long as they use disjoint plots of land, i.e., for any producers $i, j, A_i \cap A_j = \phi$ a.s.. If there is more than one producer using a given plot with non-zero input-output densities, the results obtained in the previous section might not go through since producers' decision processes are not independent and time must enter. For example, a foundation must be put in before a house is framed.

Clearly, the next step for this research is to extend the model to allow more general utility functions so that the shape and proximity of parcels may enter into utilities in a more significant manner than currently allowed. To achieve this end, Berliant and ten Raa [7] have provided a useful framework in which a topology on land parcels is proposed. However, the difficulties in the proof of existence of an equilibrium will be similar to those encountered in the previous section.

Finally, the exploration of the implications of this model for the local public goods and the producer location literature might be worthwhile. It is hoped that the combination of the model of land with a model of local public goods could reverse some of the negative results of Bewley [11] concerning local public goods.

Footnotes

1. See Berliant [4] for other properties of the underlying preferences.

2. That is because $b \in S_1$.

3. $P(x, 0)$ and $P(x, b(x))$ denote the price densities of land before and after production, respectively. Zero is the convention for no production at a point.

4. It is possible that there exists a (b', A') with $b' \neq b^*, A' \neq A^*$, such that $\int_{A'}[P(x, b'(x)) - P(x, 0) + q.b'(x)]dm(x) = \int_L[P(x, b^*(x)) - P(x, 0) + q.b^*(x)]dm(x)$

5. Berliant and ten Raa [7] use a more general utility function on the set of land parcels that allows complementarities.

6. Preferences are strictly monotone if for any (b, B, d) and $(b, B, d') \in X$, every component of d' is greater than or equal to the corresponding component of d and at least one component of d' is greater than the corresponding component of d, then $U_i(b, B, d') > U_i(b, B, d)$.

Appendix

Lemma 1: *X' and Y' are compact in the product topology on $S_1 \times L^\infty \times R^k$.*

Proof: First we show that Ω' is closed. Let $\{(b_t, g_t, y_t)\}_{t=1}^{\infty} \in \Omega'$. Suppose (b_t, g_t, y_t) converges to (b^*, g^*, y^*) in the topology defined above on $S_1 \times L^\infty \times R^k$. We need to show that $(b^*, g^*, y^*) \in \Omega'$. Given that $0 \leq g_t \leq 1$ for all t and $g_t \to g^*$ weak*ly, suppose $g^*(x) > 1$ for $x \in F$ with $m(F) > 0$. Then by weak* convergence of $\{g_t\}_{t=1}^{\infty}$, there exists a T_2 such that for all $t > T_2$, $\int_F g_t(x)dm(x) > m(F)$ which contradicts that $g_t \leq 1$. Similarly, it is false that $g^* < 0$, and we conclude that $0 \leq g^* \leq 1$. Thus, $(b^*, g^*, y^*) \in \Omega'$ and Ω' is closed.

As in the proof of Theorem 1, by Ascoli's theorem, S_1 is compact. Since we imbed G in the unit ball of L^∞, by the Banach-Alaoglu theorem, it is weak* compact. The set of feasible allocations of mobile goods is closed and bounded and, thus is compact, so Ω' is compact as a product of compact sets.

It is obvious that X' is closed given Ω' is closed and d_t is non-negative in all components. That Y' is closed follows from A1. Since X' and Y' are closed subsets of a compact set Ω', they also are compact.///

Lemma 2: *PS' is compact in the product topology on $S_3 \times S_4 \times R_+^k$.*

Proof: First, we show that PS' is closed in $S_3 \times S_4 \times R_+^k$. Let $\{(\rho_t, \rho_t', q_t)\}_{t=1}^{\infty} \subseteq PS'$ where $\rho_t \to \rho^*$ and $\rho_t' \to \rho'^*$ uniformly, and $q_t \to q^*$ componentwise. We need to show that $(\rho^*, \rho'^*, q^*) \in PS'$. By definition, for all t, $\lambda\rho_t'(x, b(x)) + \lambda'\rho_t'(x, b'(x)) = \rho_t'(x, (\lambda b + \lambda' b')(x))$ where $\lambda, \lambda' \in R, b, b' \in S_1$ and $x \in L$. Since $\rho_t' \to \rho'^*$ uniformly, it is obvious that $\lambda\rho'^*(x, b(x)) + \lambda'\rho'^*(x, b'(x)) = \rho'^*(\lambda b + \lambda' b')(x))$ for all x. By definition, $\int_L [\rho_t'(x, b(x)) + \rho_t(x)g(x)]dm(x) + \sum_{j=1}^{k} q_{j,t} = 1$. As $\rho_t \to \rho^*$ and $\rho_t' \to \rho'^*, \forall b$ and g, by arguments similar to those used in the proof of Theorem 1 and Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty}\int_L[\rho'_t(x,b(x))+\rho_t(x)g(x)]dm(x)=\int_L[\rho'^*(x,b(x))+\rho^*(x)g(x)]dm(x).$$

Since R^k_+ is closed in the usual topology on R^k,

$$\int_L[\rho'^*(x,b(x))+\rho^*(x)g(x)]dm(x)+\sum_{j=1}^{k}q^*_j=1$$

and we conclude that $(\rho^*,\rho'^*,q^*)\in PS'$. It is clear that S_3 and S_4 are bounded and equicontinuous, so by Ascoli's theorem they are compact. Thus, PS' is compact in the product topology on $S_3\times S_4\times R^k_+$.///

<u>Lemma 3</u>: *UnderA2, $\chi_i(\rho,\rho',q,I_i)$ is a closed correspondence.*

Proof: We prove $\chi_i(\rho,\rho',q,I_i)$ is a closed correspondence by showing that the graph of $\chi_i(\rho,\rho',q,I_i)$,

$$\{(\rho,\rho',q,I_i,b,g,d)\in PS'\times R_+\times X'\mid\int_L[\rho'(x,b(x))+\rho(x)g(x)]dm(x)+q.d\le I_i\},$$

is closed.

Consider two sequences, $\{(\rho_t,\rho'_t,q_t,I_t)\}_{t=1}^{\infty}$ and $\{(b_t,g_t,d_t)\}_{t=1}^{\infty}$, with $\{(\rho_t,\rho'_t,q_t,I_t)\}_{t=1}^{\infty}\subseteq PS'\times R_+$ and $\{(b_t,g_t,d_t)\}_{t=1}^{\infty}\subseteq X'$, which tend to $(\rho^*,\rho'^*,q^*,I^*)\in PS'\times R_+$ and $(b^*,g^*,d^*)\in X'$, respectively. Assume that $(b_t,g_t,d_t)\in\chi_i(\rho_t,\rho'_t,q_t,I_t)$ for every t. We need to show $(b^*,g^*,d^*)\in\chi_i(\rho^*,\rho'^*,q^*,I^*)$. Consider the following inequalities:

$$\begin{aligned}&|\rho'_t(x,b_t(x))-\rho'^*(x,b^*(x))|\\ \le&|\rho'_t(x,b_t(x))-\rho'_t(x,b^*(x))|+|\rho'_t(x,b^*(x))-\rho'^*(x,b^*(x))|\\ \le&c_3\|b_t(x)-b^*(x)\|_k+|\rho'_t(x,b^*(x))-\rho'^*(x,b^*(x))|.\end{aligned}$$

Since $\rho'_t \to \rho'^*$ and $b_t \to b^*$ uniformly, $\rho'_t(x, b_t(x)) \to \rho'^*(x, b^*(x))$ uniformly for every $x \in L$ as $t \to \infty$. As $g_t \to g^*$ weak*ly and $\rho_t \to \rho^*$ uniformly, by arguments similar to those used to prove Theorem 2, it can be established that

$$\lim_{t\to\infty} \int_L g_t(x)\rho_t(x)dm(x) = \int_L g^*(x)\rho^*(x)dm(x).$$

Since $q_t.d_t \to q^*.d^*$ as $t \to \infty$, we can show that

$$\lim_{t\to\infty} \int_L [\rho'_t(x, b_t(x)) + \rho_t(x)g_t(x)]dm(x) + q_t.d_t$$

$$= \int_L [\rho'^*(x, b^*(x)) + \rho^*(x)g^*(x)]dm(x) + q^*.d^*.$$

Since $\int_L [\rho'_t(x, b_t(x)) + \rho_t(x)g_t(x)]dm(x) + q_t.d_t \le I_t$, one has $\int_L [\rho'^*(x, b^*(x)) + \rho(x)$ $g^*(x)]dm(x) + q^*.d^* \le I^*$ and therefore, $(b^*, g^*, d^*) \in \chi_i(\rho^*, \rho'^*, q^*, I^*)$. Thus the graph of $\chi_i(\rho^*, \rho'^*, q^*, I^*)$ is closed in $PS' \times R_+ \times X'$ and χ_i is a closed correspondence.///

Lemma 4: *Let $\{(\rho_t, \rho'_t, q_t, I_t)\}_{t=1}^{\infty}$ be a sequence in the set $PS' \times R_+$ converging to $(\rho^*, \rho'^*, q^*, I^*)$ and let $(b^*, g^*, d^*) \in \chi_i(\rho^*, \rho'^*, q^*, I^*)$. Under A1, if I^* is not the minimum wealth relative to (ρ^*, ρ'^*, q^*), there exists a sequence $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$ such that $(b_t, g_t, d_t) \in \chi_i(\rho_t, \rho'_t, q_t, I_t)$ for all t and $(b_t, g_t, d_t) \to (b^*, g^*, d^*)$ as $t \to \infty$.*

Proof: Two cases have to be considered in constructing $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$ such that $(b_t, g_t, d_t) \in \chi_i(\rho_t, \rho'_t, q_t, I_t)$ for all t and $(b_t, g_t, d_t) \to (b^*, g^*, d^*)$ as $t \to \infty$.

Suppose $\int_L [\rho'^*(x, b^*(x)) + \rho^*(x)g^*(x)]\ dm(x) + q^*.d^* < I^*$. Hence there exists a T such that for $t > T$, $\int_L [\rho'_t(x, b^*(x)) + \rho_t(x)g^*(x)]\ dm(x) + q_t.d^* < I_t$. $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$ can be constructed as follows. If $t \le T, (b_t, g_t, d_t)$ is defined as any element of $\chi_i(\rho_t, \rho'_t, q_t, I_t)$. If $t > T$, $(b_t, g_t, d_t) = (b^*, g^*, d^*)$.

Suppose $\int_L [\rho'^*(x, b^*(x)) + \rho^*(x)g^*(x)]dm(x) + q^*.d^* = I^*$. Then the sequence $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$ can be constructed as follows. If $\int_L [\rho_t(x, b^*\ (x))\ + \rho_t(x)g^*(x)]dm(x)$

$+q_t.d^* \leq I_t$ for some $t > 0$, take $(b_t, g_t, d_t) = (b^*, g^*, d^*)$. Suppose $\int_L [\rho_t(x, b^*(x)) + \rho_t(x) g^*(x)]\, dm(x) + q_t.d^* > I_t > 0$ for some $t > 0$. Let

$$\lambda_t = \frac{I_t}{\int_L [\rho_t(x, b^*(x)) + \rho_t(x) g^*(x)] dm(x) + q_t d^*}.$$

It is easily seen that $1 > \lambda_t > 0$, and $\lim_{t\to\infty} \lambda_t = 1$ as $\lim_{t\to\infty} \int_L [\rho_t(x, b^*(x)) + \rho_t(x) g^*(x) dm(x) + q_t d^*] = I^*$. Now we have

$$\begin{aligned} I_t =& \lambda_t [\int_L [\rho_t(x, b^*(x)) + \rho_t(x) g^*(x)] dm(x) + q_t d^*] \\ =& \int_L [\rho_t(x, \lambda_t b^*(x)) + \rho_t(x) \lambda_t g^*(x)] dm(x) + q_t \lambda_t d^*. \end{aligned}$$

It is clear that $(\lambda_t b^*, \lambda_t g^*, \lambda_t d^*) \in X'$. Thus we can take $(b_t, g_t, d_t) = (\lambda_t b^*, \lambda_t g^*, \lambda_t d^*)$ for such t in this case. This completes the proof that such a sequence exists. ///

<u>Lemma 5</u>: *Under A2 and for $(\rho, \rho', q, I_i) \in PS' \times R_+$, $\xi_i(\rho, \rho', q, I_i)$ and $\eta(\rho, \rho', q)$ are upper hemi-continuous.*

Proof: First we show $\xi_i(\rho, \rho', q, I_i)$ is a closed correspondence. Consider two sequences, $\{(\rho_t, \rho_t', q_t, I_t)\}_{t=1}^{\infty}$ and $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$, with $\{(\rho_t, \rho_t', q_t, I_t)\}_{t=1}^{\infty} \subseteq PS' \times R_+$ and $\{(b_t, g_t, d_t)\}_{t=1}^{\infty} \subseteq X'$, which tend to $(\rho^*, \rho'^*, q^*, I^*) \in PS' \times R_+$ and $(b^*, g^*, d^*) \in X'$, respectively. Assume that $(b_t, g_t, d_t) \in \xi_i(\rho_t, \rho_t', q_t, I_t)$ for every t. We need to show $(b^*, g^*, d^*) \in \xi_i(\rho^*, \rho'^*, q^*, I^*)$.

Consider first the case $I^* \neq 0$. Since $\chi_i(\rho^*, \rho'^*, q^*, I^*)$ is closed by Lemma 3, $(b^*, g^*, d^*) \in \chi_i(\rho^*, \rho'^*, q^*, I^*)$. Suppose $(b^*, g^*, d^*) \notin \xi_i(\rho^*, \rho'^*, q^*, I^*)$ and the highest utility attained by any element of $\xi_i(\rho^*, \rho'^*, q^*, I^*)$ is $\theta' \in R$. As h_i is continuous and $g_t \to g^*$ weak*ly,

$$\lim_{t\to\infty} \int_L [h_i'(x, b_t(x)) + g_t(x) h_i(x, 0)] dm(x) = \int_L [h_i'(x, b^*(x)) + g^*(x) h_i(x, 0)] dm(x).$$

Thus, since the utility function u_i is continuous and $(b_t, g_t, d_t) \to (b^*, g^*, d^*)$,

$$\lim_{t\to\infty} U_i'(b_t, g_t, d_t) \equiv \lim_{t\to\infty} \theta_t = U_i'(b^*, g^*, d^*) = \theta^* < \theta'$$

for θ^* and $\theta_t \in R$. As shown in Lemma 4, for each $(b', g', d') \in \chi_i(\rho^*, \rho'^*, q^*, I^*)$, there exists a sequence $\{(b_t', g_t', d_t')\}_{t=1}^{\infty}$ such that $(b_t', g_t', d_t') \to (b', g', d')$, and $(b_t', g_t', d_t') \in \chi_i(\rho_t, \rho_t', q_t, I_t)$ for all t. Since $\lim_{t\to\infty} U_i'(b_t', g_t', d_t') = \theta'$, there exists a T such that for all $t > T$,

$$U_i'(b_t', g_t', d_t') > U_i'(b_t, g_t, d_t) = \theta_t.$$

Hence $(b_t, g_t, d_t) \notin \xi_i(\rho_t, \rho_t', q_t, I_t)$ as (b_t', g_t', d_t') attains higher utility for each $t > T$, which is a contradiction. Thus $(b^*, g^*, d^*) \in \xi_i(\rho^*, \rho'^*, q^*, I^*)$ and ξ_i is closed when $I^* \neq 0$.

Now consider the case that $I^* = 0$ and two sequences, $\{(\rho_t, \rho_t', q_t, I_t)\}_{t=1}^{\infty}$ and $\{(b_t, g_t, d_t)\}_{t=1}^{\infty}$, which tend to $(\rho^*, \rho'^*, q^*, 0) \in PS' \times R_+$ and $(b^*, g^*, d^*) \in X'$, respectively. Assume that $(b_t, g_t, d_t) \in \xi_i(\rho_t, \rho_t', q_t, I_t)$ for every t. We need to show $(b^*, g^*, d^*) \in \xi_i(\rho^*, \rho'^*, q^*, 0)$. Since every point in $\xi_i(\rho, \rho', q, I_i)$ is at least as good as the endowment $(0, 1_{E_i}, e_i)$ under u_i no matter $I_i = 0$ or not and u_i is continuous,

$$\lim_{t\to\infty} U_i'(b_t, g_t, d_t) \equiv \lim_{t\to\infty} \theta_t = U_i'(b^*, g^*, d^*) \geq U_i'(0, 1_{E_i}, e_i).$$

Since χ_i is closed by Lemma 3, $(b^*, g^*, d^*) \in \chi_i(\rho^*, \rho'^*, q^*, 0)$ and we conclude that $(b^*, g^*, d^*) \in \xi_i(\rho^*, \rho'^*, q^*, 0)$.

As X' is compact by Lemma 1, by Theorem 7.1.16 in Klein and Thompson [18] closedness of $\xi_i(\rho, \rho', q, I_i)$ implies that $\xi_i(\rho, \rho', q, I_i)$ is upper hemi-continuous.

Next we show that $\eta(\rho, \rho', q)$ is a closed correspondence. Consider two sequences, $\{(\rho_t, \rho_t', q_t)\}_{t=1}^{\infty}$ and $\{(b_t, 1_L, y_t)\}_{t=1}^{\infty}$, with $\{(\rho_t, \rho_t', q_t)\}_{t=1}^{\infty} \subseteq PS'$ and $\{(b_t, 1_L, y_t)\}_{t=1}^{\infty}$

$\subseteq Y'$, which tend to $(\rho^*, \rho'^*, q^*) \in PS'$ and $(b^*, 1_L, y^*) \in Y'$, respectively. Assume that $(b_t, 1_L, y_t) \in \eta(\rho_t, \rho'_t, q_t)$ for every t. We need to show $(b^*, 1_L, y^*) \in \eta(\rho^*, \rho'^*, q^*)$. Let $\int_L \rho'_t(x, b_t(x))dm(x) + q_t.y_t = M_t$ for all t and some $M_t \in R$. Since $(\rho_t, \rho'_t, q_t) \to (\rho^*, \rho'^*, q^*)$ and $(b_t, 1_L, y_t) \to (b^*, 1_L, y^*)$, one has, by arguments similar to those used to prove Lemma 3,

$$\begin{aligned}\lim_{t\to\infty} \int_L \rho'_t(x, b_t(x))dm(x) + q_t.y_t &= \int_L \rho'^*(x, b^*(x))dm(x) + q^*.y^* \\ &= \lim_{t\to\infty} M_t \equiv M^*\end{aligned}$$

for some $M^* \in R$. Suppose $(b^*, 1_L, y^*) \notin \eta(\rho^*, \rho'^*, q^*)$ and the profit attained by any element of $\eta(\rho^*, \rho'^*, q^*)$ is $M' > M^*$ where $M' \in R$. Let $(b', 1_L, y') \in \eta(\rho^*, \rho'^*, q^*)$. As the profit function is continuous, there exists a T such that for all $t > T$,

$$\int_L \rho'_t(x, b'(x))dm(x) + q_t.y' > M_t = \int_L \rho'_t(x, b_t(x))dm(x) + q_t.y_t.$$

Hence $(b_t, 1_L, y_t) \notin \eta(\rho_t, \rho'_t, q_t)$, which is a contradiction. Thus, η is a closed correspondence. Again, since Y' is compact by Lemma 1, by Theorem 7.1.16 in Klein and Thompson [18], $\eta(\rho, \rho', q)$ is upper hemi-continuous.///

Lemma 6: *$\tau(z)$ is upper hemi-continuous and compact-valued.*

Proof: First, we show $\tau(z)$ is a closed correspondence. Consider two sequences, $\{(\rho_t, \rho'_t, q_t)\}_{t=1}^{\infty}$ and $\{z_t\}_{t=1}^{\infty}$, with $\{(\rho_t, \rho'_t, q_t)\}_{t=1}^{\infty} \subseteq PS'$ and $\{z_t\}_{t=1}^{\infty} \subseteq Z$, which tend to $(\rho^*, \rho'^*, q^*) \in PS'$ and $z^* \in Z$, respectively. Assume that $(\rho_t, \rho'_t, q_t) \in \tau(z_t)$ for every t. We need to show $(\rho^*, \rho'^*, q^*) \in \tau(z^*)$.

Let $V_{z_t}(\rho_t, \rho'_t, q_t) = M_t$ for all t and some $M_t \in R$. Since $(\rho_t, \rho'_t, q_t) \to (\rho^*, \rho'^*, q^*)$ and $z_t \to z^*$, by arguments similar to those used in the proof of Lemma 3,

$$\lim_{t\to\infty} V_{z_t}(\rho_t, \rho'_t, q_t) = V_{z^*}(\rho^*, \rho'^*, q^*) = \lim_{t\to\infty} M_t \equiv M^*$$

for some $M^* \in R$. Suppose $(\rho^*, \rho'^*, q^*) \notin \tau(z^*)$ so that the value of V_{z^*} attained by any element of $\tau(z^*)$ is $M' > M^*$ where $M' \in R$. Let $(\hat{\rho}, \hat{\rho}', \hat{q}) \in \tau(z^*)$. As V_z is continuous, there exists a T such that for all $t > T, V_{z_t}(\hat{\rho}, \hat{\rho}', \hat{q}) > M_t = V_{z_t}(\rho_t, \rho'_t, q_t)$. Hence $(\rho_t, \rho'_t, q_t) \notin \tau(z_t)$, which is a contradiction. Thus, τ is a closed correspondence. Since PS' is compact by Lemma 2, $\tau(z)$ is compact-valued and by Theorem 7.1.16 in Klein and Thompson [18], $\tau(z)$ is upper hemi-continuous.///

Lemma 7: *Under A2, A4 and A5, $\xi_i(\rho, \rho', q, I_i)$ and $\eta(\rho, \rho', q)$ are convex-valued.*

Proof: Let (b, g, d) and (b', g', d') be two elements in X' with $U'_i\ (b, g, d) \geq c \in R$ and $U'_i(b', g', d') \geq c \in R$. Then if $0 \leq \lambda \leq 1$, by assumption A4,

$$\int_L [h'_i(x, (\lambda b + (1-\lambda)b')(x) + (\lambda g + (1-\lambda)g')(x)h_i(x, 0)]dm(x)$$
$$= \int_L \{\lambda[h'_i(x, b(x)) + g(x)h_i(x, 0)] + (1-\lambda)[h'_i(x, b'(x)) + g'(x)h_i(x, 0)]\}dm(x).$$

Hence,

$$U'_i(\lambda b + (1-\lambda)b', \lambda g + (1-\lambda)g', \lambda d + (1-\lambda)d')$$
$$= u_i(\int_L [h'_i(x, (\lambda b + (1-\lambda)b')(x) + (\lambda g + (1-\lambda)g')(x)h_i(x, 0)]dm(x), \lambda d + (1-\lambda)d')$$
$$= u_i(\int_L \{\lambda[h'_i(x, b(x)) + g(x)h_i(x, 0)] + (1-\lambda)[h'_i(x, b'(x)) + g'(x)h_i(x,0)]\}dm(x), \lambda d + (1-\lambda)d')$$
$$\geq \min\{u_i(\int_L [h'_i(x, b(x)) + g(x)h_i(x, 0)]dm(x), d), u_i(\int_L [h'_i(x, b'(x)) + g'(x)h_i(x, 0)]dm(x), d')\}$$
$$\geq c.$$

So the upper contour sets of U'_i are convex.

Now consider (b, g, d) and $(b', g', d') \in \chi_i(\rho, \rho', q, I_i)$, *i.e.*,

$$\int_L [\rho'(x, b(x)) + \rho(x)g(x)]dm(x) + q.d \leq I_i$$

and

$$\int_L [\rho'(x, b'(x)) + \rho(x)g'(x)]dm(x) + q.d' \leq I_i.$$

Multiplying the inequalities above by λ and $(1-\lambda)$, respectively and by Definition 10, it is straightforward to show that

$$\int_L [\rho'(x, (\lambda b + (1-\lambda)b')(x)) + \rho(x)(\lambda g + (1-\lambda)g')(x)]dm(x) + q.[\lambda d + (1-\lambda)d'] \leq I_i.$$

Hence, it is established that for $0 \leq \lambda \leq 1$, $(\lambda b + (1-\lambda)b',\ \lambda g + (1-\lambda)g',\ \lambda d + (1-\lambda)d')$ $\in \chi_i(\rho, \rho', q, I_i)$. Since the budget set correspondence $\chi_i(\rho, \rho', q, I_i)$ is convex-valued, the demand correspondence $\xi_i(\rho, \rho', q, I_i)$ is also convex-valued as it is the intersection of two convex sets, χ_i and an upper contour set of the utility function U_i.

Now we check the convexity of the iso-profit set

$$\Psi(M) = \{(b, 1_L, y) \in Y' \mid \int_L \rho'(x, b(x))dm(x) + q.y = M\}$$

for some $M \in [-\infty, \infty]$. By Definition 10, for any $(b, 1_L, y), (b',\ 1_L, y') \in \Psi(M)$,

$$\rho'(x, [\lambda b + (1-\lambda)b'](x)) = \lambda\rho'(x, b(x)) + (1-\lambda)\rho'(x, b'(x))$$

for $x \in L$. So we obtain that for $0 \leq \lambda \leq 1$,

$$\int_L \rho'(x, (\lambda b + (1-\lambda)b')(x))dm(x) + q.[\lambda y + (1-\lambda)y'] = M.$$

Thus $\Psi(M)$ is convex, and η is convex-valued as the intersection of two convex sets, $\Psi(M)$ and Y'.///

<u>Lemma 8</u>: *Let $ME(\rho, \rho', q; d, b_y) \subseteq (S_1)^N \times (L^\infty_+(L))^N$ be the set of consumers' consumption of land in a mixed equilibrium associated with $(\rho, \rho', q) \in PS'$, mobile goods allocation d and production input-output density b_y. If $\{(b_i, g_i)\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$ is an extreme point of $ME(\rho, \rho', q; d)$, then g_i is an indicator function for all i, $1 \leq i \leq N$.*

Proof: Suppsoe $\{(b_i, g_i)\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$ is an extreme point of $ME(\rho, \rho', q; d, b_y)$, but g_i is not an indicator function. Then there exists $T_i \in \mathcal{B}$ with $m(T_i) > 0$ and $\lambda_i \leq g_i(x) \leq 1 - \lambda_i$ for $x \in T_i$ where $0 < \lambda_i < \frac{1}{2}$. It is easy to show that there exist a $j \neq i$ and $1 \leq j \leq N$ such that $T_j = \{x \in L | \lambda_j \leq g_j(x) \leq 1 - \lambda_j\}$ and $m(T_i \cap T_j) > 0$ where $0 < \lambda_j < \frac{1}{2}$. Substitute $h_i(x, 0)$ and $\rho(x)$ for $h(x)$ and $P(x)$, respectively, in the lemma of Berliant [5] and follow the steps of the proof in the lemma to get the existence of r_i and $r_j \in R$ such that $h_i(x, 0) = r_i \rho(x)$ a.s., and $h_j(x, 0) = r_j \rho(x)$ a.s. for $x \in (T_i \cap T_j)$.

The remainder of the proof is to show that $\{(b_i, g_i)\}_{i=1}^N$ can be expressed as a convex combination of two elements of $ME(\rho, \rho', q; d, b_y)$, and hence could not have been an extreme point. Since $\int \rho_i(x, 0) dm(x)$ is a nonatomic measure on $T_i \cap T_j$, by Lyapunov's Theorem its range is convex. Let $\mathcal{B}_{T_i \cap T_j}$ denote the σ-algebra of measurable sets contained in $T_i \cap T_j$. We can find $M, M' \in \mathcal{B}_{T_i \cap T_j}$ such that $M \cap M' = \phi, M \cup M' = T_i \cap T_j$, and

$$\int_M \rho(x) dm(x) = \int_{M'} \rho(x) dm(x) = \frac{1}{2} \int_{T_i \cap T_j} \rho(x) dm(x)$$

Let $\epsilon' = \min(\lambda_i, \lambda_j)$. Let $\bar{b}_i(x) = b_i(x)$ and $\bar{g}_i(x) = g_i(x) - \epsilon'.1_M(x) + \epsilon'.1_{M'}(x)$ for consumer i, and $\bar{b}_j(x) = b_j(x)$ and $\bar{g}_j(x) = g_j(x) - \epsilon'.1_{M'}(x) + \epsilon'.1_M(x)$ for consumer j and $\bar{b}_k = b_k, \bar{g}_k = g_k$ for all $k \neq i, k \neq j, 1 \leq k \leq N$. It is easily seen that $\{(\bar{b}_i, \bar{g}_i, d_i)\}_{i=1}^N$ satisfies the budget constraints for all k. Next we check the utility level of consumers i

and j, and the profit of the producer. For consumer i,

$$\int_L [h_i'(x,\bar{b}_i(x)) + \bar{g}_i(x)h_i(x,0)]dm(x)$$
$$= \int_L [h_i'(x,b_i(x)) + g_i(x)h_i(x,0)]dm(x) - \epsilon'[\int_M h_i(x,0)dm(x) - \int_{M'} h_i(x,0)dm(x)]$$
$$= \int_L [h_i'(x,b_i(x)) + g_i(x)h_i(x,0)]dm(x) - r_i.\epsilon'[\int_M \rho(x)dm(x) - \int_{M'} \rho(x)dm(x)]$$
$$= \int_L [h_i'(x,b_i(x)) + g_i(x)h_i(x,0)]dm(x).$$

For consumer j,

$$\int_L [h_j'(x,\bar{b}_j(x)) + \bar{g}_j(x)h_j(x,0)]dm(x)$$
$$= \int_L [h_j'(x,b_j(x)) + g_j(x)h_j(x,0)]dm(x) - \epsilon'[\int_{M'} h_j(x,0)dm(x) - \int_M h_j(x,0)dm(x)]$$
$$= \int_L [h_j'(x,b_j(x)) + g_j(x)h_j(x,0)]dm(x) - r_j.\epsilon'[\int_{M'} \rho(x)dm(x) - \int_M \rho(x)dm(x)]$$
$$= \int_L [h_j'(x,b_j(x)) + g_j(x)h_j(x,0)]dm(x).$$

Since $\sum_{i=1}^N \bar{b}_i = \sum_{i=1}^N b_i = b_y$ and $\sum_{i=1}^N \bar{g}_i = \sum_{i=1}^N g_i = 1_L$, the producer makes the same amount of profit with the new consumption allocation $\{(\bar{b}_i, \bar{g}_i, d_i)\}_{i=1}^N$. By the fact that $\sum_{i=1}^N d_i = \sum_{i=1}^N e_i + \int_L b_y(x)dm(x)$, $\sum_{i=1}^N \bar{b}_i = \sum_{i=1}^N b_i = b_y$ and d_i is fixed for all i, the mobile goods markets are cleared. Thus, since $\{(\bar{b}_i, \bar{g}_i, d_i)\}_{i=1}^N$ maximizes each consumer's utility subject to the budget constraints and the producer maximizes his profit , $\{(\bar{b}_i, \bar{g}_i)\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$.

Now construct $\{(\hat{b}_i, \hat{g}_i)\}_{i=1}^N$ as follows. Let $\hat{b}_i(x) = b_i(x)$ and $\hat{g}_i(x) = g_i(x) + \epsilon'.1_M(x) - \epsilon'.1_{M'}(x)$ for consumer i, and $\hat{b}_j(x) = b_j(x)$ and $\hat{g}_j(x) = g_j(x) - \epsilon'.1_{M'}(x) + \epsilon'.1_M(x)$ for consumer j and $\hat{b}_k = b_k, \hat{g}_k = g_k$ for all $k \neq i, k \neq j, 1 \leq k \leq N$. By similar steps to those used above, one can demonstrate that $\{(\hat{b}_i, \hat{g}_i)\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$.

It is clear that $\frac{1}{2}(\bar{b}_i, \bar{g}_i, d_i) + \frac{1}{2}(\hat{b}_i, \hat{g}_i, d_i) = (b_i, g_i, d_i)$ and $(\bar{b}_i, \bar{g}_i, d_i) \neq (b_i, g_i, d_i)$ for all i. Hence $\{(b_i, g_i)\}_{i=1}^N$ is not an extreme point of $ME(\rho, \rho', q; d, b_y)$. This contradicts the hypothesis and g_i is an indicator function a.s. for all $i, 1 \leq i \leq N.$///

Lemma 9: *$ME(\rho, \rho', q; d, b_y)$ is convex and compact in the product topology on $(S_1)^N \times (L^\infty(L))^N$.*

Proof: Let $\{(b_i, g_i)\}_{i=1}^N$ and $\{(b_i', g_i')\}_{i=1}^N \in ME(\rho, \rho', q; d, b_y)$ and $0 \leq \lambda \leq 1$. By the definition of a mixed equilibrium, $U_i'(b_i, g_i, d_i) = U_i'(b_i', g_i', d_i) = \theta_i^*$, the highest level of utility attainable subject to the budget constraint for consumer i, which by strict monotonicity implies that

$$\int_L [h_i'(x, b_i(x)) + g_i(x)h_i(x, 0)]dm(x) = \int_L [h_i'(x, b_i'(x)) + g_i'(x)h_i(x, 0)]dm(x)$$

and for fixed d_i, by assumption A4,

$$\begin{aligned}
&U_i'(\lambda b_i + (1-\lambda)b_i', \lambda g_i + (1-\lambda)g_i', d_i) \\
=&u_i'(\int_L [h_i'(x, (\lambda b_i + (1-\lambda)b_i')(x)) + [\lambda g_i + (1-\lambda)g_i'](x)h_i(x, 0)]dm(x), d_i) \\
=&U_i'(b_i, g_i, d_i) = \theta_i^*.
\end{aligned}$$

As $\sum_{i=1}^N b_i' = \sum_{i=1}^N b_i = b_y$, it is clear that

$$\sum_{i=1}^N [\lambda b_i + (1-\lambda)b_i'] = b_y.$$

Similarly, as $\sum_{i=1}^N g_i' = \sum_{i=1}^N g_i = 1_L$, we have

$$\sum_{i=1}^N [\lambda g_i + (1-\lambda)g_i'] = 1_L.$$

Thus, as the producer's profit is determined by b_y, the producer makes the same amount of profit with the new consumption allocation $\{\lambda b_i + (1-\lambda)b_i', \lambda g_i + (1-\lambda)g_i', d_i\}_{i=1}^N$.

The mobile goods markets are cleared since $\sum_{i=1}^{N} d_i = \sum_{i=1}^{N} e_i + \int_L b_y(x)dm(x)$ and $\sum_{i=1}^{N} b_i = \sum_{i=1}^{N}[\lambda b_i + (1-\lambda)b_i'] = b_y$. Since $\forall i$

$$\int_L [\rho'(x, b_i(x)) + \rho(x)g_i(x)]dm(x) + q.d_i \leq I_i$$

and

$$\int_L [\rho'(x, b_i'(x)) + \rho(x)g_i'(x)]dm(x) + q.d_i \leq I_i,$$

it is clear that $\{(\lambda b_i + (1-\lambda)b_i', \lambda g_i + (1-\lambda)g_i', d_i)\}_{i=1}^{N}$ satisfies each consumer's budget constraint. As $\{(\lambda b_i + (1-\lambda)b_i', \lambda g_i + (1-\lambda)g_i', d_i)\}_{i=1}^{N}$ maximizes each consumer's utility subject to the budget constraint and the producer maximizes his profit, it satisfies the definition of a mixed equilibrium and belongs to $ME(\rho, \rho', q; d, b_y)$. Thus, $ME(\rho, \rho', q; d, b_y)$ is convex.

Next we prove $ME(\rho, \rho', q; d, b_y)$ is compact in the product topology on $(S_1)^N \times (L_+^\infty)^N$. Let $\{(b_1^t, g_1^t), ..., (b_N^t, g_N^t)\} \in ME(\rho, \rho', q; d, b_y)$ for all $t > 0$ and suppose $b_i^t \to b_i$ uniformly and $g_i^t \to g_i$ weak*ly for each $i = 1, ..., N$. Note that for every t, $U_i'(b_i^t, g_i^t, d_i) = \theta_i^*$. By definition of weak* convergence, that $\rho \in L^1(L)$, and Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty} \int_L [\rho'(x, b_i^t(x)) + g_i^t(x)\rho(x)]dm(x) = \int_L [\rho'(x, b_i(x)) + g_i(x)\rho(x)]dm(x),$$

so $\{(b_1, g_1), ..., (b_N, g_N)\}$ is in the budget set for every $i = 1, ..., N$. For consumers, we need only check whether $U_i'(b_i, g_i, d_i) = \lim_{t\to\infty} U_i'\ (b_i^t, g_i^t, d_i)$.

$$\begin{aligned}
U_i'(b_i, g_i, d_i) &= u_i(\int_L [h_i'(x, b_i(x)) + g_i(x)h_i(x, 0))]dm(x), d_i) \\
&= u_i(\lim_{t\to\infty} \int_L [h_i'(x, b_i^t(x)) + g_i^t(x)h_i(x, 0)]dm(x), d_i) \\
&= \lim_{t\to\infty} u_i(\int_L [h_i'(x, b_i^t(x)) + g_i^t(x)h_i(x, 0)]dm(x), d_i) \\
&= \lim_{t\to\infty} \theta_i^* = \theta_i^*.
\end{aligned}$$

Hence (b_i, g_i, d_i) attains the highest level of utility attainable subject to the budget constraint for consumer i. It is straightforward to check $\sum_{i=1}^{N} g_i(x) = 1$, a.s. $(x \in L)$ and $\sum_{i=1}^{N} b_i = b_y$ given $\sum_{i=1}^{N} g_i^t(x) = 1$, a.s. and $\sum_{i=1}^{N} b_i^t = b_y$ for all t. Thus b_y is not changed and the producer is making the same amount of profit with the consumption allocation in the limit. The mobile goods markets are cleared since $\sum_{i=1}^{N} d_i = \sum_{i=1}^{N} e_i + \int_L b_y(x)dm(x)$ and $\sum_{i=1}^{N} b_i^t = \sum_{i=1}^{N} b_i = b_y$. Hence $\{(b_i, g_i)\}_{i=1}^{N} \in ME(\rho, \rho', q; d, b_y)$ and $ME(\rho, \rho', q; d, b_y)$ is a closed set in the product topology on $(S_1)^N \times (L^\infty(L))^N$. Since the consumption set is compact by lemma 1, $ME(\rho, \rho', q; d, b_y)$ is also compact./ / /

Lemma 10: *Under A6, if $\{(b_i, g_i)\}_{i=1}^{N} \in ME(\rho, \rho', q; d, b_y)$ is an extreme point of $ME(\rho, \rho', q; d, b_y)$, then, for all i, $g_i(x) > 0$ a.s. whenever $b_i(x) \neq 0$ a.s..*

Proof: We need to show that if $g_j(x) = 0 a.s.$ for some j and $x \in L$, then $b_j(x) = 0a.s.$. Let $B_i = \{x \in L | g_i(x) > 0\}$. If $m(B_i) = 0$ for some i, then it does not matter whether other consumers have nonzero input-output desity on B_i. So consider $m(B_i) \neq 0$. Since $\{(b_i, g_i)\}_{i=1}^{N}$ is an extreme point of $ME(\rho, \rho', q; d, b_y)$, by Lemma 8, g_i is an indicator function, i.e., consumer i owns B_i and $g_j(x) = 0a.s.$ for $x \in B_i$ and $j \neq i$, while consumer j owns B_j. Suppose $b_j(x) \neq 0$ for $x \in B_{ij} \subseteq B_i$ with $m(B_{ij}) > 0$. Let U_i and U_j be the utility functions for consumers i and j, respectively and

$$\bar{\gamma}_i \equiv \sup\{\gamma_i \in R | U_i^0(b_i, B_i, d_i)h_i(x, 0) \geq \gamma_i \rho(x) a.s., x \in B_{ij}\}$$

and

$$\underline{\gamma}_j \equiv \inf\{\gamma_j \in R | U_j^0(b_j, B_j, d_j)h_j(x, 0) \leq \gamma_j \rho(x) a.s., x \in B_{ij}\}.$$

Since U_i^0, h_i and ρ are continuous functions defined on compact sets and $h_i(x, 0) > 0a.s.$, $h_i(x, 0)$ and $\rho(x)$ are bounded for $x \in B_i$ and in equilibrium, $\rho > 0a.s.$. Thus it is clear that $0 < \bar{\gamma}_i < \infty$, $0 < \underline{\gamma}_j < \infty$ and for $x \in B_{ij}$,

$$\frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{\bar{\gamma}_i} \geq \rho(x) \geq \frac{U_j^0(b_j, B_j, d_j)h_j(x,0)}{\underline{\gamma}_j}. \tag{7}$$

By assumption A6 (1), $\forall s > 0, U_i^0(b_i, B_i, d_i)h_i(x,0) \neq sU_j^0(b_j, B_j, d_j)h_j(x,0)$ a.s., so (7) is actually

$$\frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{\bar{\gamma}_i} > \frac{U_j^0(b_j, B_j, d_j)h_j(x,0)}{\underline{\gamma}_j} a.s.(x \in B_{ij}). \tag{8}$$

Let $\lambda_i \in R$ and

$$L(b_i, B_i, d_i) = U_i(b_i, B_i, d_i) - \lambda_i(\int_{B_i} P(x, b_i(x))dm(x) + q.d_i - I_i).$$

Recall that mobile good l is assumed to be essential to every consumer in A6 (2). Differentiate $L(b_i, B_i, d_i)$ with respect to $d_{i,l}$ and set the derivative to zero to get the first-order condition, $U_i^l(b_i, B_i, d_i) = \lambda_i q_l$ where q_l is the price of mobile good l. Similarly, there is a $\lambda_j \in R$ such that $U_j^l(b_j, B_j, d_j) = \lambda_j q_l$.

Since consumer i owns B_i, we need to show that $\bar{\gamma}_i \geq \lambda_i$ and $\underline{\gamma}_j \leq \lambda_j$. If $\lambda_i > \bar{\gamma}_i$, then consumer i can be better off by exchanging some land in B_{ij} for equal value of mobile good l. Formally, let $\epsilon > 0$ and

$$B_\epsilon = \{x \in B_{ij} | \lambda_i > \bar{\gamma}_i + \epsilon > \frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{\rho(x)} \geq \bar{\gamma}_i\}.$$

$\epsilon(\alpha)$ is defined to be such that $\int_{B_{\epsilon(\alpha)}} \rho(x)dm(x) = \alpha q_l$. By allowing small marginal change in land allocations which has equal value as some amount of mobile good l, we will show that the utility of consumer i can be increased. Let $B_i' = B_i / B_{\epsilon(\alpha)}$, $d_{i,l}' = d_{i,l} + \alpha$, and $d_i' = (d_{i,1}, d_{i,2}, ..., d_{i,l}', ..., d_{i,k})$. Thus (b_i, B_i', d_i') is in the budget set for consumer i since

$$\int_{B_i'} \rho(x)dm(x) + d_i'q = \int_{B_i} \rho(x)dm(x) - \int_{B_{\epsilon(\alpha)}} \rho(x)dm(x) + d_iq + \alpha q_l$$

$$= \int_{B_i} \rho(x)dm(x) + d_i q.$$

Using Taylor's series to expand U_i around (b_i, B_i, d_i) and evaluate it at (b_i, B_i', d_i'), we have

$$\begin{aligned}
& U_i(b_i, B_i', d_i') - U_i(b_i, B_i, d_i) \\
&= U_i^0(b_i, B_i, d_i)\Big(\int_{B_i'} h(x,0)dm(x) - \int_{B_i} h(x,0)dm(x)\Big) + U_i^l(b_i, B_i, d_i)(d_{i,l}' - d_{i,l}) + \mathcal{R} \\
&= -U_i^0(b_i, B_i, d_i)\int_{B_{\epsilon(\alpha)}} h(x,0)dm(x) + U_i^l(b_i, B_i, d_i).\alpha + \mathcal{R},
\end{aligned}$$

where $\mathcal{R}$ is the remainder. Since $\mathcal{R}$ converges to 0 at a rate faster than

$$-U_i^0(b_i, B_i, d_i)\int_{B_{\epsilon(\alpha)}} h(x,0)dm(x) + U_i^l(b_i, B_i, d_i).\alpha$$

converges to zero as $\alpha \to 0$, we choose $\alpha > 0$ such that for all $\alpha' \leq \alpha$,

$$| - U_i^0(b_i, B_i, d_i)\int_{B_{\epsilon(\alpha)}} h(x,0)dm(x) + U_i^l(b_i, B_i, d_i).\alpha'| > |\mathcal{R}|.$$

As $\lambda_i > \bar{\gamma}_i + \epsilon > \frac{U_i^0(b_i,B_i,d_i)h_i(x,0)}{\rho(x)} \geq \bar{\gamma}_i$ for $x \in B_{\epsilon(\alpha)}$ and $U_i^l(b_i, B_i, d_i) = \lambda_i q_l$, we have

$$\begin{aligned}
& - U_i^0(b_i, B_i, d_i)\int_{B_{\epsilon(\alpha)}} h(x,0)dm(x) + U_i^l(b_i, B_i, d_i)\alpha \\
> & - \lambda_i \int_{B_{\epsilon(\alpha)}} \rho(x)dm(x) + \alpha\lambda_i q_l \\
= & [\lambda_i - \lambda_i]\alpha q_l \\
= & 0,
\end{aligned}$$

hence for $\alpha' < \alpha$, $U_i(b_i, B_i', d_i') > U_i(b_i, B_i, d_i)$, a contradiction. Therefore $\bar{\gamma}_i \geq \lambda_i$. Analogously we can show that $\underline{\gamma}_j \leq \lambda_j$, as otherwise consumer j can be better off by giving up some of mobile good l and buying equal value of land in B_{ij}. Now by (8)

$$\begin{aligned}
\frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{\lambda_i} &\geq \frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{\bar{\gamma}_i} \\
&> \frac{U_j^0(b_j, B_j, d_j)h_j(x,0)}{\underline{\gamma}_j} \geq \frac{U_j^0(b_j, B_j, d_j)h_j(x,0)}{\lambda_j} a.s. (x \in B_{ij}).
\end{aligned}$$

Since the utility function is strictly monotone in mobile goods, $q_l > 0$ in the equilibrium for all l and we have

$$\frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{\lambda_i q_l} > \frac{U_j^0(b_j, B_j, d_j)h_j(x,0)}{\lambda_j q_l}.$$

Since $U_i^l(b_i, B_i, d_i) = \lambda_i q_l$ and $U_j^l(b_j, B_j, d_j) = \lambda_j q_l$,

$$\frac{U_i^0(b_i, B_i, d_i)h_i(x,0)}{U_i^l(b_i, B_i, d_i)} > \frac{U_j^0(b_j, B_j, d_j)h_j(x,0)}{U_j^l(b_j, B_j, d_j)}. \tag{9}$$

By (9) and assumption A6 (3), we obtain

$$\frac{U_i^0(b_i, B_i, d_i)|\frac{\partial h_i^l(x,b_i(x))}{\partial b_{i,l}(x)}|}{U_i^l(b_i, B_i, d_i)} > \frac{U_j^0(b_j, B_j, d_j)|\frac{\partial h_j^l(x,b_j(x))}{\partial b_{j,l}(x)}|}{U_j^l(b_j, B_j, d_j)}. \tag{10}$$

As $b_j(x) \neq 0$ for $x \in B_{ij}$ in the equilibrium, $U_j^0(b_j, B_j, d_j)|\frac{\partial h_j^l(x,b_j(x))}{\partial b_{j,l}(x)}|$ has to equal $U_j^l(b_j, B_j, d_j)$ for $x \in B_{ij}$, otherwise the utility of consumer j is not maximized as in the equilibrium marginal cost of mobile good l equals marginal cost of changing input-output density in the direction of mobile good l. (This is a consequence of the competitive behavior of the firm.) Similarly, we have

$$\begin{cases} U_i^0(b_i, B_i, d_i)|\frac{\partial h_i^l(x,b_i(x))}{\partial b_{i,l}(x)}| = U_i^l(b_i, B_i, d_i) \text{ if } b_i(x) \neq 0 \\ U_i^0(b_i, B_i, d_i)|\frac{\partial h_i^l(x,b_i(x))}{\partial b_{i,l}(x)}| \leq U_i^l(b_i, B_i, d_i) \text{ if } b_i(x) = 0. \end{cases}$$

Thus for all $b_i(x)$, $x \in B_i$,

$$U_i^0(b_i, B_i, d_i)|\frac{\partial h_i^l(x, b_i(x))}{\partial b_{i,l}(x)}| \leq U_i^l(b_i, B_i, d_i),$$

and by (10), we obtain the contradiction that

$$1 \geq \frac{U_i^0(b_i, B_i, d_i)|\frac{\partial h_i^l(x,b_i(x))}{\partial b_{i,l}(x)}|}{U_i^l(b_i, B_i, d_i)} > \frac{U_j^0(b_j, B_j, d_j)|\frac{\partial h_j^l(x,b_j(x))}{\partial b_{j,l}(x)}|}{U_j^l(b_j, B_j, d_j)} = 1.$$

Hence the hypothesis is false, and $b_j(x) = 0$ on B_i for $j \neq i$.///

References

1. W. Alonso, "Location and Land Use," Harvard University Press, Cambridge, 1964.

2. M. Beckmann, On the distribution of urban rent and residential density, *J. Econ. Theory* **1** (1969), 60-67.

3. M. Berliant, A characterization of the demand for land, *J. Econ. Theory* **33** (1984), 289-300.

4. M. Berliant, Equilibrium models with land : A criticism and an alternative, *Reg. Sci. Urban Econ.* **15** (1985), 325-340.

5. M. Berliant, An equilibrium existence result for an economy with land, *J. Math. Econ.* **14** (1985), 53-56.

6. M. Berliant, A utility representation for a preference relation on a $\sigma-$algebra, *Econometrica* **54** (1986), 359-362.

7. M. Berliant and T. ten Raa, "On the Continuum Approach of Spatial and Some Local Public Goods or Product Differentiation Models," unpublished manuscript, University of Rochester, 1986.

8. M. Berliant and T. ten Raa, "A Foundation of Location Theory : Consumer Preferences and Demand," unpublished manuscript, University of Rochester, 1986.

9. M. Berliant, Y. Y. Papageorgiou, and P. Wang, "On Welfare Theory and Urban Economics," RCER Working Paper # 95, University of Rochester, 1987.

10. T. Bewley, Existence of equilibrium in economies with infinitely many cmmodities, *J. Econ. Theory* **4** (1972), 514-540.

11. T. Bewley, A critique of Tiebout's theory of local public expenditures, *Econometrica* **49** (1981), 713-740.

12. G. Debreu, Valuation equilibrium and pareto optimum, *Proc. Nat. Acad. Sci.* **40** (1954), 588-592.

13. G. Debreu, "Theory of Value," Yale University Press, New Haven, 1959.

14. K. Fan, Fixed point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci.* **39** (1952), 121-126.

15. C. Goffman, "Real Functions," Prindle, Weber & Schmidt, Boston, 1970.

16. W. Hildenbrand, "Core and Equilibria of a Large Economy," Princeton University Press, Princeton, 1974.

17. L. Jones, A competitive model of commodity differentiation, *Econometrica* **52** (1984), 507-530.

18. E. Klein and A. Thompson, "Theory of Correspondences," Canadian Mathematical Society, Series of Monographs and Advanced Texts, John Wiley and Sons, New York, 1984.

19. J. Munkres, "Topology: A First Course," Prentice-Hall, Englewood Cliffs, 1975.

20. W. Rudin, "Functional Analysis," McGraw-Hill, New York, 1973.

21. W. Wheaton, Monocentric models of urban land use : Contributions and criticisms, in "Current Issues in Urban Economics" (P. Mieszkowski and M. Straszheim, Eds.), Johns Hopkins, Baltimore, 1979.

Rochester Center for Economic Research
University of Rochester
Department of Economics
Rochester, NY 14627

1986-87 DISCUSSION PAPERS

WP#33 OIL PRICE SHOCKS AND THE DISPERSION HYPOTHESIS, 1900 - 1980
by Prakash Loungani, January 1986

WP#34 RISK SHARING, INDIVISIBLE LABOR AND AGGREGATE FLUCTUATIONS
by Richard Rogerson, (Revised) February 1986

WP#35 PRICE CONTRACTS, OUTPUT, AND MONETARY DISTURBANCES
by Alan C. Stockman, October 1985

WP#36 FISCAL POLICIES AND INTERNATIONAL FINANCIAL MARKETS
by Alan C. Stockman, March 1986

WP#37 LARGE-SCALE TAX REFORM: THE EXAMPLE OF EMPLOYER-PAID HEALTH INSURANCE PREMIUMS
by Charles E. Phelps, March 1986

WP#38 INVESTMENT, CAPACITY UTILIZATION AND THE REAL BUSINESS CYCLE
by Jeremy Greenwood and Zvi Hercowitz, April 1986

WP#39 THE ECONOMICS OF SCHOOLING: PRODUCTION AND EFFICIENCY IN PUBLIC SCHOOLS
by Eric A. Hanushek, April 1986

WP#40 EMPLOYMENT RELATIONS IN DUAL LABOR MARKETS (IT'S NICE WORK IF YOU CAN GET IT!)
by Walter Y. Oi, April 1986

WP#41 SECTORAL DISTURBANCES, GOVERNMENT POLICIES, AND INDUSTRIAL OUTPUT IN SEVEN EUROPEAN COUNTRIES
by Alan C. Stockman, April 1986

WP#42 SMOOOTH VALUATIONS FUNCTIONS AND DETERMINANCY WITH INFINITELY LIVED CONSUMERS
by Timothy J. Kehoe, David K. Levine and Paul R. Romer, April 1986

WP#43 AN OPERATIONAL THEORY OF MONOPOLY UNION-COMPETITIVE FIRM INTERACTION
by Glenn M. MacDonald and Chris Robinson, June 1986

WP#44 JOB MOBILITY AND THE INFORMATION CONTENT OF EQUILIBRIUM WAGES: PART 1, by Glenn M. MacDonald, June 1986

WP#45 SKI-LIFT PRICING, WITH APPLICATIONS TO LABOR AND OTHER MARKETS
by Robert J. Barro and Paul M. Romer, May 1986, revised April 1987

WP#46 FORMULA BUDGETING: THE ECONOMICS AND ANALYTICS OF FISCAL POLICY UNDER RULES, by Eric A. Hanushek, June 1986

WP#48 EXCHANGE RATE POLICY, WAGE FORMATION, AND CREDIBILITY by Henrik Horn and Torsten Persson, June 1986

WP#49 MONEY AND BUSINESS CYCLES: COMMENTS ON BERNANKE AND RELATED LITERATURE, by Robert G. King, July 1986

WP#50 NOMINAL SURPRISES, REAL FACTORS AND PROPAGATION MECHANISMS by Robert G. King and Charles I. Plosser, Final Draft: July 1986

WP#51 JOB MOBILITY IN MARKET EQUILIBRIUM by Glenn M. MacDonald, August 1986

WP#52 SECRECY, SPECULATION AND POLICY by Robert G. King, (revised) August 1986

WP#53 THE TULIPMANIA LEGEND by Peter M. Garber, July 1986

WP#54 THE WELFARE THEOREMS AND ECONOMIES WITH LAND AND A FINITE NUMBER OF TRADERS, by Marcus Berliant and Karl Dunz, July 1986

WP#55 NONLABOR SUPPLY RESPONSES TO THE INCOME MAINTENANCE EXPERIMENTS by Eric A. Hanushek, August 1986

WP#56 INDIVISIBLE LABOR, EXPERIENCE AND INTERTEMPORAL ALLOCATIONS by Vittorio U. Grilli and Richard Rogerson, September 1986

WP#57 TIME CONSISTENCY OF FISCAL AND MONETARY POLICY by Mats Persson, Torsten Persson and Lars E. O. Svensson, September 1986

WP#58 ON THE NATURE OF UNEMPLOYMENT IN ECONOMIES WITH EFFICIENT RISK SHARING, by Richard Rogerson and Randall Wright, September 1986

WP#59 INFORMATION PRODUCTION, EVALUATION RISK, AND OPTIMAL CONTRACTS by Monica Hargraves and Paul M. Romer, September 1986

WP#60 RECURSIVE UTILITY AND THE RAMSEY PROBLEM by John H. Boyd III, October 1986

WP#61 WHO LEAVES WHOM IN DURABLE TRADING MATCHES by Kenneth J. McLaughlin, October 1986

WP#62 SYMMETRIES, EQUILIBRIA AND THE VALUE FUNCTION by John H. Boyd III, December 1986

WP#63 A NOTE ON INCOME TAXATION AND THE CORE by Marcus Berliant, December 1986

WP#64 INCREASING RETURNS, SPECIALIZATION, AND EXTERNAL ECONOMIES: GROWTH AS DESCRIBED BY ALLYN YOUNG, By Paul M. Romer, December 1986

WP#65 THE QUIT-LAYOFF DISTINCTION: EMPIRICAL REGULARITIES by Kenneth J. McLaughlin, December 1986

WP#66 FURTHER EVIDENCE ON THE RELATION BETWEEN FISCAL POLICY AND THE TERM STRUCTURE, by Charles I. Plosser, December 1986

WP#67 INVENTORIES AND THE VOLATILITY OF PRODUCTION by James A. Kahn, December 1986

WP#68 RECURSIVE UTILITY AND OPTIMAL CAPITAL ACCUMULATION, I: EXISTENCE, by Robert A. Becker, John H. Boyd III, and Bom Yong Sung, January 1987

WP#69 MONEY AND MARKET INCOMPLETENESS IN OVERLAPPING-GENERATIONS MODELS, by Marianne Baxter, January 1987

WP#70 GROWTH BASED ON INCREASING RETURNS DUE TO SPECIALIZATION by Paul M. Romer, January 1987

WP#71 WHY A STUBBORN CONSERVATIVE WOULD RUN A DEFICIT: POLICY WITH TIME-INCONSISTENT PREFERENCES by Torsten Persson and Lars E.O. Svensson, January 1987

WP#72 ON THE CONTINUUM APPROACH OF SPATIAL AND SOME LOCAL PUBLIC GOODS OR PRODUCT DIFFERENTIATION MODELS by Marcus Berliant and Thijs ten Raa, January 1987

WP#73 THE QUIT-LAYOFF DISTINCTION: GROWTH EFFECTS by Kenneth J. McLaughlin, February 1987

WP#74 SOCIAL SECURITY, LIQUIDITY, AND EARLY RETIREMENT by James A. Kahn, March 1987

WP#75 THE PRODUCT CYCLE HYPOTHESIS AND THE HECKSCHER-OHLIN-SAMUELSON THEORY OF INTERNATIONAL TRADE by Sugata Marjit, April 1987

WP#76 NOTIONS OF EQUAL OPPORTUNITIES by William Thomson, April 1987

WP#77 BARGAINING PROBLEMS WITH UNCERTAIN DISAGREEMENT POINTS by Youngsub Chun and William Thomson, April 1987

WP#78 THE ECONOMICS OF RISING STARS by Glenn M. MacDonald, April 1987

WP#79 STOCHASTIC TRENDS AND ECONOMIC FLUCTUATIONS by Robert King, Charles Plosser, James Stock, and Mark Watson, April 1987

WP#80 INTEREST RATE SMOOTHING AND PRICE LEVEL TREND-STATIONARITY
by Marvin Goodfriend, April 1987

WP#81 THE EQUILIBRIUM APPROACH TO EXCHANGE RATES
by Alan C. Stockman, revised, April 1987

WP#82 INTEREST-RATE SMOOTHING
by Robert J. Barro, May 1987

WP#83 CYCLICAL PRICING OF DURABLE LUXURIES
by Mark Bils, May 1987

WP#84 EQUILIBRIUM IN COOPERATIVE GAMES OF POLICY FORMULATION
by Thomas F. Cooley and Bruce D. Smith, May 1987

WP#85 RENT SHARING AND TURNOVER IN A MODEL WITH EFFICIENCY UNITS OF HUMAN CAPITAL
by Kenneth J. McLaughlin, revised, May 1987

WP#86 THE CYCLICALITY OF LABOR TURNOVER: A JOINT WEALTH MAXIMIZING HYPOTHESIS
by Kenneth J. McLaughlin, revised, May 1987

WP#87 CAN EVERYONE BENEFIT FROM GROWTH? THREE DIFFICULTIES
by Herve' Moulin and William Thomson, May 1987

WP#88 TRADE IN RISKY ASSETS
by Lars E.O. Svensson, May 1987

WP#89 RATIONAL EXPECTATIONS MODELS WITH CENSORED VARIABLES
by Marianne Baxter, June 1987

WP#90 EMPIRICAL EXAMINATIONS OF THE INFORMATION SETS OF ECONOMIC AGENTS
by Nils Gottfries and Torsten Persson, June 1987

WP#91 DO WAGES VARY IN CITIES? AN EMPIRICAL STUDY OF URBAN LABOR MARKETS
by Eric A. Hanushek, June 1987

WP#92 ASPECTS OF TOURNAMENT MODELS: A SURVEY
by Kenneth J. McLaughlin, July 1987

WP#93 ON MODELLING THE NATURAL RATE OF UNEMPLOYMENT WITH INDIVISIBLE LABOR
by Jeremy Greenwood and Gregory W. Huffman

WP#94 TWENTY YEARS AFTER: ECONOMETRICS, 1966-1986
by Adrian Pagan, August 1987

WP#95 ON WELFARE THEORY AND URBAN ECONOMICS
by Marcus Berliant, Yorgos Y. Papageorgiou and Ping Wang, August 1987

WP#96 ENDOGENOUS FINANCIAL STRUCTURE IN AN ECONOMY WITH PRIVATE INFORMATION
by James Kahn, August 1987

WP#97 THE TRADE-OFF BETWEEN CHILD QUANTITY AND QUALITY: SOME EMPIRICAL EVIDENCE
by Eric Hanushek, September 1987

WP#98 SUPPLY AND EQUILIBRIUM IN AN ECONOMY WITH LAND AND PRODUCTION
by Marcus Berliant and Hou-Wen Jeng, September 1987

WP#99 AXIOMS CONCERNING UNCERTAIN DISAGREEMENT POINTS FOR 2-PERSON BARGAINING PROBLEMS
by Youngsub Chun, September 1987

To order copies of the above papers complete the attached invoice and return to Christine Massaro, W. Allen Wallis Institute of Political Economy, RCER, 109B Harkness Hall, University of Rochester, Rochester, NY 14627. Three (3) papers per year will be provided free of charge as requested below. Each additional paper will require a $5.00 service fee which must be enclosed with your order. For your convenience an invoice is provided below in order that you may request payment from your institution as necessary. Please make your check payable to the **Rochester Center for Economic Research**. **Checks must be drawn from a U.S. bank and in U.S. dollars**.

W. Allen Wallis Institute for Political Economy

Rochester Center for Economic Research, Working Paper Series

OFFICIAL INVOICE

Requestor's Name ______________________________

Requestor's Address ______________________________

____

____

____

Please send me the following papers free of charge (**Limit: 3 free per year**).

WP# ______ WP# ______ WP# ______

I understand there is a $5.00 fee for each additional paper. Enclosed is my check or money order in the amount of $__________. Please send me the following papers.

WP# ______ WP# ______ WP# ______

WP# ______ WP# ______ WP# ______

WP# ______ WP# ______ WP# ______

WP# ______ WP# ______ WP# ______