Axioms Concerning Uncertain Disagreement Points for 2-Person Bargaining Problems

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AXIOMS CONCERNING UNCERTAIN DISAGREEMENT POINTS FOR 2-PERSON BARGAINING PROBLEMS

by

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1. Introduction

Suppose bargaining takes place today, without the precise knowledge of the location of the disagreement point, this uncertainty being resolved tomorrow. Under what conditions will agents reach an agreement today? The minimal requirement is that each agent should be guaranteed at least the minimum of what he/she receives when the uncertainty is lifted tomorrow. Otherwise, the agent is definitely better off by waiting until tomorrow. We require that all agents should be guaranteed at least this minimum. This requirement of *disagreement point quasi-concavity* was introduced in Chun and Thomson [1987b] (variants of which are studied by Chun and Thomson [1987a], Livne [1986b], Peters [1986b] and Peters and van Damme [1987]). The purpose of this paper is to explore the implication of this axiom for 2-person bargaining problems.

To that purpose, we introduce a new family of solutions, which we call *linear solutions*. They are defined as follows. Let δ be a function associating with each problem a non-negative direction such that all interior points of the feasible set on the line passing through the disagreement point in the direction assigned by the function δ have the same direction. Then the *linear solution relative to* δ is defined by choosing as solution outcome of each problem the maximal feasible point such that the vector of utility gains from the disagreement point is in the direction determined by applying δ to the problem. This family of solutions, which we call the *linear family*, is fairly large, including many well-known solutions such as the Nash and egalitarian solutions. It also includes the lexicographic egalitarian and Kalai-Rosenthal solutions.

By imposing disagreement point quasi-concavity in conjunction with the standard conditions of weak Pareto optimality, individual rationality and continuity, we characterize continuous members of the linear family. Also, by strengthening weak Pareto optimality and weakening continuity, we characterize the Pareto optimal members of the linear family. Other characterizations of the family can be obtained by using axioms related to disagreement point quasi-concavity. We also show how well-known subfamilies or elements of the family can be singled out by imposing additional axioms.

The methodology, which we adopt here, is the axiomatic approach to bargaining theory, as introduced by Nash [1950]. However, the focus on the formulation of the bargaining problem is different. In the traditional formulation, it is typically assumed that the disagreement point is fixed. The possibility of varying disagreement points has recently been the object of a number of studies (Thomson [1987], Livne [1986a], Peters [1986b], and others). Moreover, bargaining situations in which the feasible set is known but the disagreement point is uncertain have been studied extensively (Chun and Thomson [1987a,b], Livne [1986b] and Peters and van Damme [1987]). The present paper is also focused on the role of uncertain disagreement points in bargaining.

The paper is organized as follows. Section 2 contains some preliminaries and introduces the basic axioms. Section 3 states our main axiom of disagreement point quasi-concavity, and characterizes the linear family. Section 4 discusses axioms related to disagreement point quasi-concavity and establishes alternative characterizations of the linear family. Finally, section 5 characterizes various subfamilies including the egalitarian, lexicographic egalitarian, Nash and Kalai-Rosenthal solutions.

2. Preliminaries

A 2-person bargaining problem, or simply a *problem,* is a pair (S, d), where S is a subset of $\Re^2$ and d is a point in S, such that

(1) S is convex and closed,

(2) $a_i(S) \equiv max\{x_i | x \equiv (x_1, x_2) \in S\}$ exists for $i = 1, 2$,

(3) S is *comprehensive,* i.e., for all $x \in S$ and for all $y \in \Re^2$, if $y \leq x$, [1] then $y \in S$,

(4) there exists $x \in S$ with $x > d$.

S is the *feasible set.* Each point x of S is a *feasible alternative.* The coordinates of x are the von Neumann-Morgenstern utility levels attained by the agents through the choice of some joint action. d is the *disagreement point* (or status quo). The intended interpretation of (S, d) is as follows: the agents can achieve any point of S if they unanimously agree on it. If they do not agree on any point, they end up at d. Let Σ^2 be the class of all problems and Γ^2 be the class of all feasible sets satisfying (1), (2) and (3).

A *solution* is a function $F: \Sigma^2 \rightarrow \Re^2$ such that for all $(S, d) \in \Sigma^2, F(S, d) \in S$. $F(S, d)$, the value taken by the solution F when applied to the problem (S, d), is called the *solution outcome of* (S, d).

The following axioms, which are standard in the literature, will be adopted whenever necessary.

Weak Pareto Optimality (WPO). For all $(S, d) \in \Sigma^2$ and for all $x \in \Re^2$, if $x > F(S, d)$, then $x \notin S$.

Pareto Optimality (PO). For all $(S, d) \in \Sigma^2$ and for all $x \in \Re^2$, if $x \geq F(S, d)$, then $x \notin S$.

Let $WPO(S) \equiv \{x \in S |$ for all $x' \in \Re^2, x' > x$ implies $x' \notin S\}$ be the set of *weakly Pareto optimal points of* S. Similarly, let $PO(S) \equiv \{x \in S |$ for all $x' \in \Re^2, x' \geq x$ implies $x' \notin S\}$ be the set of *Pareto optimal points of S.*

[1] Vector inequalities: given $x, y \in \Re^n, x \geqq y, x \geq y, x > y$.

Individual Rationality (IR). For all $(S, d) \in \Sigma^2$, $F(S, d) \geqq d$.

Let $IR(S, d) \equiv \{x \in S | x \geqq d\}$ be the set of *individually rational points of* (S, d).

d-Continuity (d-CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S, d) \in \Sigma^2$, if $S^k = S$ for all k and $d^k \to d$, then $F(S^k, d^k) \to F(S, d)$.

In the following, convergence of a sequence of sets is evaluated in the Hausdorff topology.

S-Continuity (S-CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S, d) \in \Sigma^2$, if $S^k \to S$ and $d^k = d$ for all k, then $F(S^k, d^k) \to F(S, d)$.

WPO requires that there be no feasible alternative at which all agents are better off than at the solution outcome. PO requires that the solution outcome should exhaust all gains from cooperation. IR requires that no agent be worse off at the solution outcome than at the disagreement point. Finally, d-CONT (respectively, S-CONT) requires that a small change in the disagreement point (respectively, the feasible set) cause only a small change in the solution outcome.

The following notation and terminology will be used frequently. Given $x_1, \ldots, x_k \in \Re^n$, $comp\{x_1, \ldots, x_k\}$ is the *comprehensive hull* of these points (the smallest comprehensive set containing them). Given $A \subset \Re^n$, $Int(A)$ is the relative interior of A. Δ^{n-1} is the (n-1)-dimensional simplex. Given $x \in \Re^2$ and $\delta \in \Delta^1$, $\ell(x, \delta)$ is the line passing through x in the direction δ. Finally, given $x, y \in \Re^2$ such that $x \neq y$, $\ell(x, y)$ is the line passing through x and y.

3. Disagreement Point Quasi-Concavity. The Main Characterization

The main purpose of this paper is to explore the implication of the following axiom, introduced by Chun and Thomson [1987b], for 2-person bargaining problems.

Disagreement Point Quasi-Concavity (D.Q-CAV). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$, for all i and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$, then

$$F_i(S, \alpha d^1 + (1-\alpha)d^2) \geqq min\{F_i(S, d^1), F_i(S, d^2)\}.$$

(Note that $(S, \alpha d^1 + (1-\alpha)d^2)$ is a well-defined element of Σ^2.)

This axiom can be motivated on the basis of timing of bargaining. Consider agents *today*, who, *tomorrow*, will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. The agents have two options: either they wait until tomorrow for the uncertainty to be lifted and solve then whatever problem has come up, or they consider the problem obtained by taking as disagreement point the average of d^1 and d^2 and solve that problem today. If, for some agent i, $F_i(S, \frac{d^1+d^2}{2})$ is smaller than the minimum of $F_i(S, d^1)$ and $F_i(S, d^2)$, then the agent will definitely prefer waiting until the uncertainty is lifted. For agent i to be persuaded that the problem should be solved today, he should be guaranteed at least the minimum of $F_i(S, d^1)$ and $F_i(S, d^2)$. Imposing D.Q-CAV provides this minimum incentive to all agents.

We are interested in the following new family of solutions, which generalizes the egalitarian, lexicographic egalitarian, Nash and Kalai-Rosenthal solutions.

Definition. Let $\delta : \Sigma^2 \to \Delta^1$ be a function such that for all $S \in \Gamma^2$ and for all $y \in Int(S)$, $y \in \ell(d, \delta(S, d))$ implies that $\delta(S, y) = \delta(S, d)$. The *linear solution relative to* δ, F^δ, is defined by setting, for each $(S, d) \in \Sigma^2$, $F^\delta(S, d)$ equal to $\ell(d, \delta(S, d)) \cap WPO(S)$.

Note that, for the solution F^δ to be well-defined, it should be that for all $S \in \Gamma^2$ (i) for all $d^1, d^2 \in Int(S)$, if $\delta(S, d^1) \neq \delta(S, d^2)$, then $\ell(d^1, \delta(S, d^1)) \cap \ell(d^2, \delta(S, d^2)) \cap Int(S) = \emptyset$, and (ii) $\delta(S, \cdot)$ is continuous with respect to d.

We now turn to the results. The proof of Lemma 1 is the same as the proof of Lemma 1 in Chun and Thomson [1987b].

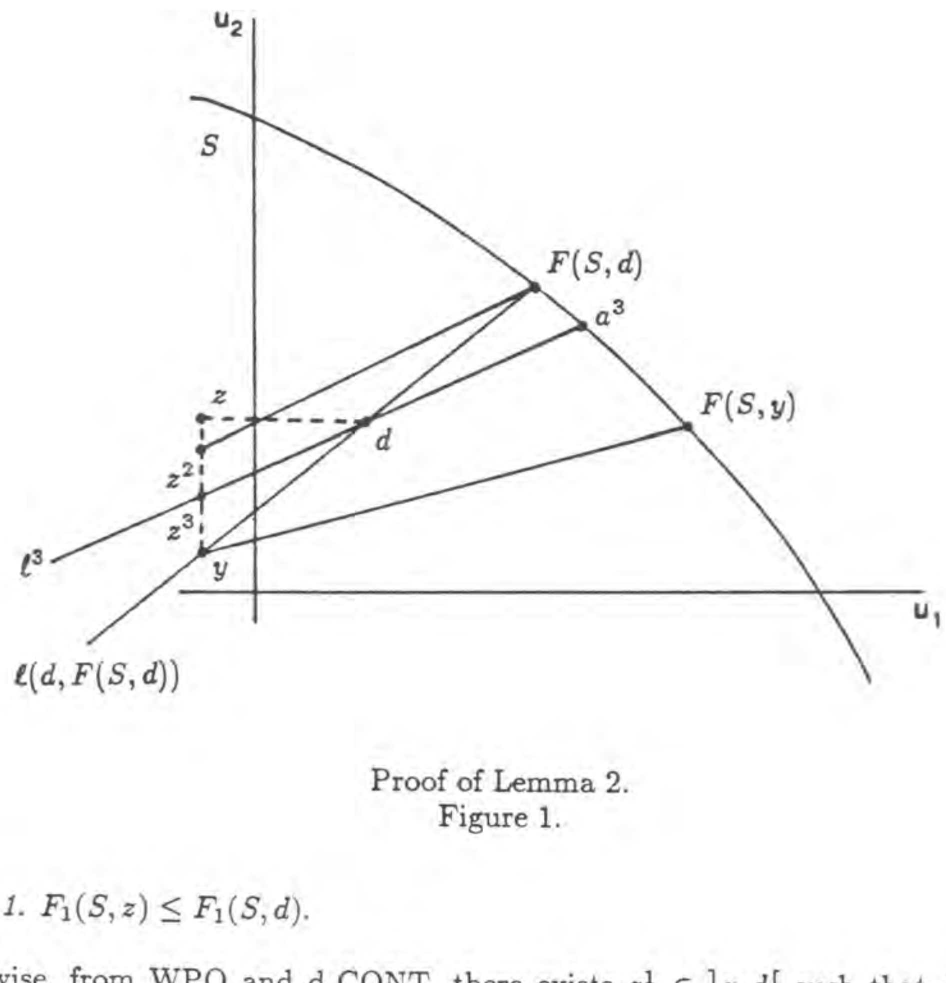
Lemma 1. *Let F be a solution satisfying WPO, IR and D.Q-CAV. Also let $(S, d) \in \Sigma^2$ be such that $F(S, d) \in PO(S)$. Then for all $x \in [d, F(S, d)[$, $F(S, x) = F(S, d)$.*

Proof. First, note that $(S, x) \in \Sigma^2$ for all $x \in [d, F(S, d)[$. Let $x \in]d, F(S, d)[$ be given. Let $\bar{\lambda} \in]0, 1[$ be such that $x = \bar{\lambda} d + (1 - \bar{\lambda}) F(S, d)$, and $\{\lambda^k\} \subset]0, 1[$ be such that $\lambda^k < \bar{\lambda}$ for all k and $\lambda^k \to \bar{\lambda}$. Also, let $x^k \equiv \frac{x - \lambda^k d}{1 - \lambda^k}$ for all k. Note that $(S, x^k) \in \Sigma^2$ for all k. By D.Q-CAV, $F_i(S, x) \geqq min\{F_i(S, x^k), F_i(S, d)\}$ for all i and for all k. As $k \to \infty$, $x^k \to F(S, d)$ and since $F(S, d) \in PO(S)$, it follows from IR that $F(S, x^k) \to F(S, d)$. Therefore, we obtain $F(S, x) \geqq F(S, d)$. Since $F(S, d) \in PO(S)$, we conclude that $F(S, x) = F(S, d)$. Q.E.D.

Lemma 2. *Let F be a solution satisfying WPO, IR, d-CONT and D.Q-CAV. Also, let $(S, d) \in \Sigma^2$ be such that $F(S, d) \in Int(PO(S))$. Then for all $x \in \ell(d, F(S, d)) \cap Int(S)$, $F(S, x) = F(S, d)$.*

Proof. Let F and $(S, d) \in \Sigma^2$ satisfying the hypothesis of Lemma 2 be given. From Lemma 1, we know that for all $x \in [d, F(S, d)[$, $F(S, x) = F(S, d)$. Now suppose, by way of contradiction, that there exists $y \in Int(S)$ such that $d \in]y, F(S, d)[$ and $F(S, y) \neq F(S, d)$. Since $F(S, d) \in Int(PO(S))$, it follows from WPO and d-CONT, we can assume that $F(S, y) \in Int(PO(S))$.

(a) We consider the case when $\ell(d, F(S,d))$ is neither horizontal nor vertical. Suppose that $F_1(S,y) > F_1(S,d)$. Let $z \equiv (y_1, d_2)$.



Proof of Lemma 2.
Figure 1.

Claim 1. $F_1(S,z) \le F_1(S,d)$.

Otherwise, from WPO and d-CONT, there exists $z^1 \in \,]z,d[$ such that $F_1(S,d) < F_1(S,z^1) \le F_1(S,z)$ and $F(S,z^1) \in PO(S)$. From Lemma 1, for all $x \in [z^1, F(S,z^1)[$, $F(S,x) = F(S,z^1)$. Since $F_2(S,z^1) \ge z_2^1 = d_2$ by IR, there exists an $\bar{x} \in [d, F(S,d)[\cap [z^1, F(S,z^1)[$, which is a contradiction.

Claim 2. $F_1(S,y) > F_1(S,d)$ is impossible.

Since $F_1(S,z) \leq F_1(S,d)$, by d-CONT, there exists $z^2 \in [z,y[$ such that $F(S,z^2) = F(S,d)$. From Lemma 1, for all $x \in [z^2, F(S,d)[$, $F(S,x) = F(S,d)$. Also, from WPO, IR and Lemma 1, we have for all $x \in [z^2,d]$, $F(S,x) = F(S,d)$. Now define the sequence of problems $\{(S,z^k)\}$ by $z^{k+1} \equiv \frac{1}{2}(z^k + y)$ for all $k \geq 2$. Also, for all $k \geq 3$, let ℓ^k be the line passing through z^k and d, and $a^k \equiv \ell^k \cap WPO(S)$. For all $x \in]z^2,z^3]$, if $F_1(S,d) < F_1(S,x) \leqq min\{F_1(S,y), a_1^3\}$, then there exists z' such that $z' \in \ell(x, F(S,x)) \cap \ell(z^2,d)$. Since we assumed that $F(S,x) \neq F(S,d)$, this is impossible. Therefore, for all $x \in]z^2,z^3]$, we have $F_1(S,x) \leqq F_1(S,d)$ or $F_1(S,x) > min\{F_1(S,y), a_1^3\}$. By d-CONT, we have $F_1(S,x) \leq F_1(S,d)$ for all $x \in [z^2,z^3]$. By repeating the same procedure, for all $x \in [z^2,y[$, we obtain $F_1(S,x) \leqq F_1(S,d)$. Therefore, $F_1(S,y) > F_1(S,d)$ contradicts d-CONT.

By a similar argument, we obtain a contradiction to $F_1(S,y) < F_1(S,d)$.

(b) Now suppose that $\ell(d, F(S,d))$ is horizontal and there exists $y \in Int(S)$ such that $d \in]y, F(S,d[$ and $F(S,y) \neq F(S,d)$. By IR and d-CONT, there exists $z^1 \in [y,d[$ such that $F(S,z^1) \neq F(S,d)$ and that $\ell(z^1, F(S,z^1))$ is positively sloped. From (a), for all $z \cap \ell(z^1, F(S,z^1)) \cap Int(S)$, $F(S,z) = F(S,z^1)$. Now let a^* be the Pareto optimal point of S on the line passing through d parallel to $\ell(z^1, F(S,z^1))$. For some $z \in [z^1,d[$, say z^2, if $\ell(z^2, F(S,z^2))$ is flatter than $\ell(z^1, F(S,z^1))$, then there exists $z' \in Int(S)$ such that $z' \in \ell(z^1, F(S,z^1)) \cap \ell(z^2, F(S,z^2))$, which is impossible. Therefore, for all $z \in [z^1,d[$, $F_1(S,z) < a_1^*$. This is incompatible with d-CONT. A similar argument can be established when $\ell(d, F(S,d))$ is vertical. Q.E.D.

Remark 1. Lemma 1 can easily be generalized to n-person problems. However, it remains an open question whether Lemma 2 can be generalized to such problems.

Now we present our main results.

Theorem 1. *A solution satisfies PO, IR, d-CONT and D.Q-CAV if and only if it is a linear solution* F^δ *with the additional property, that for all* $(S,d) \in \Sigma^2$, $\ell(d, \delta(S,d)) \cap WPO(S) \backslash PO(S) = \emptyset$.

Proof. It is obvious that all F^δ satisfy IR, d-CONT and D.Q-CAV, and if δ satisfies the additional property, PO. Conversely, let F be a solution satisfying the four axioms. For all $(S,d) \in \Sigma^2$, let $\delta(S,d) \equiv \frac{F(S,d)-d}{||F(S,d)-d||}$. Since PO and IR together imply that $F(S,d) \geq d$, δ is a well-defined function from Σ^2 to Δ^1. It is enough to show that for all $(S^1,d^1),(S^2,d^2) \in \Sigma^2$, if $S^1 = S^2 \equiv S$ and $d^2 \in \ell(d^1, F(S,d^1))$, then $\delta(S,d^2) = \delta(S,d^1)$. If $F(S,d^1) \in Int(PO(S))$, then the desired conclusion follows from Lemma 2. Suppose now that $F(S,d^1) \notin Int(PO(S))$ and that $\delta(S,d^1) \neq \delta(S,d^2)$. From Lemma 1, for all $d \in [d^1, F(S,d^1)[$, $F(S,d) = F(S,d^1)$ and for all $d \in [d^2, F(S,d^2)[$, $F(S,d) = F(S,d^2)$. By PO and d-CONT, there exists $d' \in]d^1, d^2[$ such that $F(S,d') \in Int(PO(S))$, $F(S,d') \neq F(S,d^2)$ and that either $\ell(d', F(S,d')) \cap [d^1, F(S,d^1)[\neq \emptyset$ or $\ell(d', F(S,d')) \cap [d^2, F(S,d^2)[\neq \emptyset$. Since $F(S,d') \neq F(S,d^1)$ and $F(S,d') \neq F(S,d^2)$, it is a contradiction.

Finally, we note that PO implies that, for all $(S,d) \in \Sigma^2$, $\ell(d,\delta(S,d)) \cap$ $WPO(S) \backslash PO(S) = \emptyset$. Q.E.D.

Remark 2. The family of solutions characterized in Theorem 1 is fairly large, including the Nash, Kalai-Rosenthal and lexicographic egalitarian solutions. However, the egalitarian solution is excluded, since it violates PO.

Theorem 2. *A solution satisfies WPO, IR, d-CONT, S-CONT and D.Q-CAV if and only if it is a linear solution F^δ with the additional property, that $\delta(\cdot, x)$ be continuous with respect to S.*

Proof. It is obvious that all F^δ satisfy WPO, IR, d-CONT and D.Q-CAV, and if $\delta(\cdot, x)$ is continuous with respect to S, S-CONT. Conversely, let F be a solution satisfying the five axioms. For all $(S, d) \in \Sigma^2$, let $\delta(S, d) \equiv \frac{F(S,d)-d}{||F(S,d)-d||}$. Since WPO and IR together imply that $F(S, d) \geq d$, δ is a well-defined function from Σ^2 to Δ^1. It is enough to show that, for all $(S, d) \in \Sigma^2$, if there exists $d' \in \ell(d, F(S, d)) \cap Int(S)$, then $\delta(S, d') = \delta(S, d)$. If $F(S, d) \in Int(PO(S))$, then the desired conclusion follows from Lemma 2. Otherwise, let $\{(S^k, d)\} \subset \Sigma^2$ be a sequence of problems such that for all k, $F(S^k, d) \in Int(PO(S^k))$ and $d \in Int(S^k)$ and such that $S^k \to S$. By the previous argument, $F(S^k, d) = F^\delta(S^k, d)$ for all k, and by S-CONT, $F(S, d) = F^\delta(S, d)$.

Finally, we note that S-CONT implies the continuity of $\delta(\cdot, x)$ with respect to S in the Hausdorff topology. Q.E.D.

Remark 3. The family of solutions characterized in Theorem 2 is fairly large, including the Nash, egalitarian and Kalai-Rosenthal solutions. However, the lexicographic egalitarian solution is excluded, since it violates S-CONT.

4. Variants of the Main Result

Recently, bargaining situations in which the feasible set is known but the disagreement point is uncertain have been studied extensively. Several axioms related to disagreement point quasi-concavity have appeared. Here we discuss how the linear family can be characterized using these axioms.

The first axiom, which we call *weak disagreement point linearity,* [2] was introduced by Livne [1986b] in his study of the Nash solution.

Weak Disagreement Point Linearity (W.D.LIN). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^n$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$ and $F(S, d^1) = F(S, d^2) \equiv x$, then $F(S, \alpha d^1 + (1-\alpha)d^2) = x$.

Again, this axiom can be motivated on the basis of timing of bargaining. Suppose agents *today,* who, *tomorrow,* will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. Suppose that the solution outcomes of the two problems are coincide. Since all agents receive the same amount tomorrow irrespective of the uncertainty, it is natural to require that they should receive the same amount today. Imposing W.D.LIN on the solutions makes the uncertainty not affect the final allocation.

Now we explore the implication of this axiom for 2-person bargaining problems. By replacing D.Q-CAV by W.D.LIN in Theorems 1 and 2, we obtain the same conclusions. In addition, by using the following weak condition, a characterization of the linear family can be established.

Boundary (BOUND). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S, d) \in \Sigma^2$, if $S^k = S$ for all k, $F(S, d) = x$ and $d^k \to x$, then $F(S^k, d^k) \to x$.

For a solution satisfying Pareto optimality, BOUND is just a considerable weakening of IR. For a solution satisfying only weak Pareto optimality, BOUND is a weak continuity property requiring that if the disagreement point is closer to the boundary of the

[2] He calls this property 'Independence of Convex Combination of Equivalent Conflict Outcomes.'

feasible set, then the solution outcome is also closer to the disagreement point. It is a very weak condition satisfied by all well-known solutions.

Now we have the following result.

Lemma 3. *Let F be a solution satisfying WPO, IR, d-CONT, BOUND and W.D.LIN. Also, let $(S, d) \in \Sigma^2$ be given. Then for all $x \in [d, F(S,d)[$, $F(S,x) = F(S,d)$.*

Proof. First, note that for all $x \in [d, F(S,d)[$, $(S,x) \in \Sigma^2$. We assume that $WPO(S)$ contains a vertical segment. The case that $WPO(S)$ contains a horizontal (or both vertical and horizontal) segment can be dealt with similarly. Now suppose, by way of contradiction, that there exists $d^1 \in [d, F(S,d)[$ such that $F(S,d^1) \neq F(S,d)$. Two cases are possible:

(i) $F_2(S,d^1) > F_2(S,d)$.

Note that if $F_2(S,d^1) > F_2(S,d)$, IR implies that $\ell(d, F(S,d))$ is not vertical. Let $d^2 \in Int(S)$ be such that $d_1^2 = d_1$ and that for all $a \in IR(S,d^1)$, $a_2 > F_2(S,d^1)$. By WPO, $F(S,d^2) \in WPO(S)$ and by IR, $F_2(S,d^2) > F_2(S,d^1)$. By d-CONT, there exists $d^3 \in [d^2, d[$ such that $F(S,d^3) = F(S,d^1)$. By W.D.LIN, for all $d' \in [d^1, d^3]$, $F(S,d') = F(S,d^1)$.

Now let $d(\lambda)$ be a parametrization of $[d^1, F(S,d)]$ such that $d(0) = d^1$ and $d(1) = F(S,d)$. By d-CONT, $F(S,d(\lambda))$ moves continuously. By BOUND, there exists $\bar{\lambda} \in [0,1[$ such that $F_2(S,d^1) > F_2(S,d(\bar{\lambda})) \geqq F_2(S,d)$. Let $d(\bar{\lambda}) = d^4$. Also, by d-CONT, there exists $d^5 \in [d^3, d]$ such that $F(S,d^5) = F(S,d^4)$. By W.D.LIN, for all $d' \in [d^4, d^5]$, $F(S,d') = F(S,d^4)$. Then $[d^1, d^3]$ and $[d^4, d^5]$ intersect. Let d^6 be the intersection point. Clearly, $d^6 \in Int(S)$. Since $F(S,d^1) \neq F(S,d^4)$, it is a contradiction.

(ii) $F_2(S, d^1) < F_2(S, d)$.

From the same argument as in (i), we have, for all $d' \in [d^1, F(S, d^1)[$, $F_2(S, d') \leqq F_2(S, d^1)$. Let d^2 be a point in $]d^1, F(S, d^1)[$.

Let $d(\lambda)$ be a parametrization of $[d^2, F(S, d)]$ such that $d(0) = d^2$ and $d(1) = F(S, d)$. By d-CONT, $F(S, d(\lambda))$ moves continuously. By BOUND, there exists $\bar{\lambda} \in [0, 1[$ such that $F_2(S, d) \geqq F_2(S, d(\bar{\lambda})) > F_2(S, d^1)$. Let $d(\bar{\lambda}) = d^3$. Also, by d-CONT, there exists $d^4 \in [d^2, d]$ such that $F(S, d^4) = F(S, d^3)$. By W.D.LIN, for all $d' \in [d^3, d^4]$, $F(S, d') = F(S, d^3)$. Then $[d^1, F(S, d^1)[$ and $[d^3, d^4]$ intersect. Let d^5 be the intersection point. Clearly, $d^5 \in Int(S)$. Since $d^5 \in [d^1, F(S, d^1)[$, $F_2(S, d^5) \leqq F_2(S, d^1)$, and since $d^5 \in [d^3, d^4]$, $F_2(S, d^5) = F_2(S, d^3) > F_2(S, d^1)$. This is a contradiction. Q.E.D.

Theorem 3. *A solution satisfies WPO, IR, d-CONT, BOUND and W.D.LIN if and only if it is a linear solution.*

Proof. It is obvious that all F^δ satisfy the five axioms. Conversely, let F be a solution satisfying the five axioms. First, we know from Lemma 3 that for all $(S, d) \in \Sigma^2$, and for all $x \in [d, F(S, d)[$, $F(S, x) = F(S, d)$. Now we extend the conclusion of Lemma 3 to all $x \in \ell(d, F(S, d)) \cap Int(S)$. Since the proof is similar to that of Lemma 2, we omit it. Q.E.D.

The second axiom was introduced by Peters and van Damme [1987] [3] in their study of the Nash solution.

[3] They call this property 'convexity.'

Disagreement Point Linearity (D.LIN). For all $(S, d) \in \Sigma^2$ and for all $\alpha \in]0, 1]$, $F(S, \alpha d + (1-\alpha)F(S,d)) = F(S, d)$.

This is a strengthening of W.D.LIN to require that, for a given problem (S, d), a new problem obtained by taking the same feasible set and a different disagreement point, which is a convex combination of the old disagreement point and its solution outcome, should yield the same solution outcome. If we extend our domain of bargaining problems to allow the disagreement point to lie on the boundary of the feasible set, and define the solution outcome of such problems be the disagreement point, then the motivation similar to W.D.LIN can be given.

Now we explore the implication of this axiom for 2-person bargaining problems. Again, by replacing D.Q-CAV by D.LIN in Theorems 1 and 2, we obtain the same conclusion. In addition, the following theorem can be established.

Theorem 4. *A solution satisfies WPO, IR, d-CONT and D.LIN if and only if it is a linear solution.*

Proof. It is obvious that all F^δ satisfy the four axioms. The converse statement is established by exploiting the logical implications between D.LIN, W.D.LIN and BOUND. Indeed, it can easily be shown that (i) WPO and D.LIN together imply W.D.LIN, and that (ii) d-CONT and D.LIN together imply BOUND. Therefore, by Theorem 3, we obtain the desired conclusion. Q.E.D.

Remark 4. If IR in the Theorem 4 is dropped from the list, then following family of the *generalized linear solutions* can be characterized. Let $B^1 \equiv \{x \in \Re^2 | \sum |x_i| = 1$ and $-x \notin \Re^2_+\}$ and given $x \in \Re^2$ and $\delta \in B^1$, let $\bar{\ell}(d, \delta)$ be the line passing through d

in the direction δ. Also, given $(S, d) \in \Sigma^2$, let $\bar{\ell}(d, \delta) \cap WPO(S)$ be the weakly Pareto optimal point of S on the half-line passing through d in the direction δ.

Definition. Let δ be a function such that, for all $(S, d) \in \Sigma^2$, $\delta(S, d) \in B^1$ and that, for all $S \in \Gamma^2$ and for all $y \in Int(S)$, $y \in \bar{\ell}(d, \delta(S, d))$ implies that $\delta(S, y) = \delta(S, d)$ and that $\delta(S, \cdot)$ is continuous with respect to d. Given the function δ, the *generalized linear solution relative to* δ is defined by setting, for each $(S, d) \in \Sigma^2$, $F^\delta(S, d)$ equal to $\bar{\ell}(d, \delta(S, d)) \cap WPO(S)$.

5. Further Characterizations

In this section, we impose additional axioms (or strengthen the axioms used in the Theorems 1 and 2) to characterize important subfamilies of the linear family.

5.1. Egalitarian Solution

First, we consider a subfamily of the linear solutions, which generalizes the well-known egalitarian solution (Kalai [1977] and Thomson and Myerson [1980]).

Definition. Given a continuous function $\delta : \Gamma^2 \to \Delta^1$, the *directional solution relative to* δ, E^δ, is defined by setting, for all $(S, d) \in \Sigma^2$, $E^\delta(S, d)$ equal to $\ell(d, \delta(S)) \cap WPO(S)$. Given $\alpha \in \Delta^1$, the *weighted egalitarian solution with weights* α, E^α, is defined by setting, for all $(S, d) \in \Sigma^2$, $E^\alpha(S, d)$ equal to $\ell(d, \alpha) \cap WPO(S)$. The *egaltarian solution* is obtained by choosing $\alpha_1 = \alpha_2$.

This family can be characterized by the following axiom, which strengthens D.Q-CAV.

Disagreement Point Concavity (D.CAV). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all $\alpha \in [0, 1]$, if $S^1 = S^2 \equiv S$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) \geqq \alpha F(S, d^1) + (1-\alpha)F(S, d^2).$$

This axiom, introduced and studied in Chun and Thomson [1987a], gives an even greater incentive to all agents to reach an agreement today than D.Q-CAV does. Consider agents *today*, who, *tomorrow*, will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. The agents have two options: either they wait until tomorrow for the uncertainty to be lifted and solve then whatever problem has come up, or they consider the problem obtained by taking as disagreement point the average of d^1 and d^2 and solve that problem today. The expected payoff associated with the first option is $\frac{F(S,d^1)+F(S,d^2)}{2}$ and that associated with the second option is $F(S, \frac{d^1+d^2}{2})$, since $\frac{d^1+d^2}{2}$ is the corresponding "expected"disagreement point. If either $F(S, \frac{d^1+d^2}{2})$ weakly dominates $\frac{F(S,d^1)+F(S,d^2)}{2}$ or the reverse holds, all agents agree on when to do. A conflict may arise if neither of these inequalities holds. Imposing D.CAV on the solutions prevents such a conflict.

It can easily be checked that D.CAV implies D.Q-CAV. D.CAV can be regarded as a dual to an axiom considered by Myerson [1981] concerning uncertainty in the feasible set (variants of which are studied by Perles and Maschler [1981] and Peters [1986a]).

The following results, which can be generalized to n-person bargaining problems, are due to Chun and Thomson [1987a]. We note that d-CONT is not used.

Theorem 5. *A solution satisfies WPO, IR, S-CONT and D.CAV if and only if it is a directional solution.*

The family of weighted egalitarian solution can be characterized by strengthening IR to the following axiom.

Independence of Non-Individually Rational Alternatives (INIR). For all $(S, d) \in \Sigma^2$, $F(S, d) = F(comp\{IR(S, d)\}, d)$.

This axiom, introduced by Peters [1986a], says that the non-individually rational alternatives are irrelevant to the determination of the solution outcome. It is a natural condition since agents are guaranteed their utilities at d. It can easily be checked that WPO, INIR and S-CONT (or PO and INIR) together imply IR.

Theorem 6. *A solution satisfies WPO, INIR, S-CONT and D.CAV if and only if it is a weighted egalitarian solution.*

The egalitarian solution is the only weighted egalitarian solution satisfying the following axiom.

Symmetry (SY). For all $(S, d) \in \Sigma^2$ and for all permutations $\pi : \{1, 2\} \to \{1, 2\}$, if $S = \pi(S)$ and $d = \pi(d)$, then $F_1(S, d) = F_2(S, d)$.

Sy says that if the only information available on the conflict situation is contained in the mathematical description of (S, d), and (S, d) is a symmetric problem, then there is no ground for favoring one agent at the expense of another.

Corollary 1. *A solution satisfies WPO, INIR, S-CONT, D.CAV and SY if and only if it is the egalitarian solution.*

5.2. Lexicographic Egalitarian Solution

The directional solutions often violate PO. The following extension, called the *lexicographic egalitarian solution*, is an adaptation of the egalitarian solution that satisfies PO. However, we note that this solution does not satisfy S-CONT.

Definition. The *lexicographic egalitarian solution, L,* is defined by setting, for all $(S,d) \in \Sigma^2$, $L(S,d) = E(S,d)$ if $E(S,d) = PO(S)$ and $L(S,d) = \{x \in PO(S) | x_1 = E_1(S,d)$ or $x_2 = E_2(S,d)\}$, otherwise. [4]

Chun and Thomson [1987a] showed that there is no solution satisfying PO, IR and D.CAV. However, the following slight weakening of D.CAV is compatible with PO and IR.

Restricted Disagreement Point Concavity (R.D.CAV). For all $(S^1,d^1),(S^2,d^2) \in \Sigma^2$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$ and $F(S,d^1), F(S,d^2) \in Int(PO(S))$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) \geqq \alpha F(S,d^1) + (1-\alpha)F(S,d^2).$$

The motivation for this axiom is same as for D.CAV, except that the conclusion is required to hold for the interior of the Pareto optimal set. For a solution satisfying PO, if it chooses the boundary point of the Pareto optimal set as the solution outcome, then the solution outcome becomes less sensitive to changes in the disagreement point. Therefore, it is unreasonable to require that the solution behave well even on the boundary. It can easily be checked that PO, d-CONT and R.D.CAV (or PO, IR and R.D.CAV) together imply D.Q-CAV.

To characterize the lexicographic egalitarian solution, S-CONT is weakened to the following condition.

[4] This solution has been studied by Imai [1983] and Lensberg [1982].

Pareto-Continuity (P-CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S, d) \in \Sigma^2$, if $S^k \to S$, $PO(S^k) \to PO(S)$ and $d^k = d$ for all k, then $F(S^k, d^k) \to F(S, d)$.

P-CONT requires that a small change in the feasible set and the Pareto optimal set causes only a small change in the solution outcome. It can easily be checked that S-CONT implies P-CONT.

Now we characterize the lexicographic egalitarian solution. The proof is similar to that of Theorems 1 and 6 of Chun and Thomson [1987a], which characterize the egalitarian solution. We note again that PO and INIR together imply IR.

Lemma 4. *Let F be a solution satisfying PO, IR, d-CONT and R.D.CAV. Also let $S \in \Gamma^2$ and $d^1, d^2 \in Int(S)$. If $\alpha F(S, d^1) + (1-\alpha) F(S, d^2) \in Int(PO(S))$ for all $\alpha \in [0, 1]$, then the line $\ell(d^1, F(S, d^1))$ is parallel to the line $\ell(d^2, F(S, d^2))$.*

Proof. Let S, d^1 and d^2 be as in the Lemma. By Theorem 1, for all $y \in \ell(d^1, F(S, d^1)) \cap Int(S)$, $F(S, y) = F(S, d)$. Let $d^3 \in \ell(d^1, F(S, d^1)) \cap Int(S)$ be such that $d^3 \neq d^1$. Without loss of generality, suppose that $d^3 \in \left]d^1, F(S, d^1)\right[$. Now let $z^i \equiv \frac{1}{2}(d^i + d^2)$ for $i = 1, 3$. By R.D.CAV, we have

$$F(S, z^i) \geqq \frac{1}{2}\{F(S, d^i) + F(S, d^2)\} = \frac{1}{2}\{F(S, d^1) + F(S, d^2)\} \equiv x \quad \text{for} \quad i = 1, 3.$$

Since $x \in PO(S)$, we have $F(S, z^i) = x$ for $i = 1, 3$. For all $z \in \ell(z^1, z^2) \cap Int(S)$ such that $z^3 = \lambda z^1 + (1-\lambda) z$ for some $\lambda \in \left]0, 1\right[$, by R.D.CAV and $\lambda < 1$, we have $x \geqq F(S, z)$. Since $x \in Int(PO(S))$, we have, by PO, $F(S, z) = x$. By IR, $\ell(z^1, z^2)$ passes through x. This is possible only if $\ell(d^1, F(S, d^1))$ is parallel to $\ell(d^2, F(S, d^2))$.

Q.E.D.

Lemma 5. *Let F be a solution satisfying PO, IR, d-CONT, P-CONT and R.D.CAV.*

Also let $S \in \Gamma^2$ be given. Then for all $(S,d) \in \Sigma^2$ such that $F(S,d) \in Int(PO(S))$, $\ell(d, F(S,d))$ has the same slope.

Proof. Let $S \in \Gamma^2$ be a polygonal feasible set such that $Int(PO(S)) \neq \emptyset$. Let $\{S^i | i \in I\}$, where $I \subseteq N$ and $S^i \equiv \{x \in \Re^2 | \sum p^i_j x_j \leqq c^i$ for some $p^i \in \Delta^1$ and $c^i \in \Re\}$ be a minimal collection such that $S = \cap_{i \in I} S^i$. Let $i \in I$ be such that $p^i > 0$. By PO and IR, there exists $x \in Int(S)$ such that $F(S,x) \in PO(S^i) \cap Int(PO(S))$. By Lemma 4, for all $d \in Int(S)$, if $F(S,d) \in PO(S^i) \cap Int(PO(S))$, then the line $\ell(x, F(S,x))$ is parallel to the line $\ell(d, F(S,d))$. Let the common direction be denoted by $\delta(S^i)$. By IR, $\delta(S^i) \geq 0$. Also, for all $d \in Int(S)$, if $\ell(d, \delta(S^i)) \cap PO(S^i) \equiv y \in Int(PO(S))$, $F(S,d) = y$ from Lemma 4.

Now let S^i and S^j be such that $PO(S^i) \cap PO(S^j) \neq \emptyset$. Note that $a \in PO(S^i) \cap PO(S^j)$ implies that $a \in Int(PO(S))$. Without loss of generality, suppose that $i = 1$ and $j = 2$. We claim that $\delta(S^1) = \delta(S^2)$. Let $a \in PO(S^1) \cap PO(S^2)$, $y^1 \in PO(S^1) \cap Int(PO(S))$, $d^1 \equiv y^1 - \delta(S^1)$ and $d^2 \equiv a - \delta(S^2)$. By the previous step, $F(S, d^1) = y^1$ and $F(S, d^2) = a$. By Lemma 4 applied to d^1 and d^2, we conclude that $\ell(d^1, F(S, d^1))$ is parallel to $\ell(d^2, F(S^2)) = \ell(d^2, a)$. Therefore, we have $\delta(S^1) = \delta(S^2)$. Repeating the argument, we have $\delta(S^i) = \delta(S^1)$ for all S^i with $p^i > 0$, as desired. Q.E.D.

Lemma 6. *Let F be a solution satisfying PO, INIR, d-CONT, P-CONT and R.D.CAV. Then for all $(S,d) \in \Sigma^2$ such that $F(S,d) \in Int(PO(S))$, $\ell(d, F(S,d))$ has the same slope.*

Proof. Let $S^1, S^2 \in \Gamma^2$ be such that $Int(PO(S^1)) \cap Int(PO(S^2)) \neq \emptyset$. For $i = 1, 2$, let $\delta(S^i)$ be the direction derived in the proof of Lemma 5. Also, let $T \equiv S^1 \cap S^2$. Since $Int(PO(S^i)) \neq \emptyset$, there exists $d^i \in Int(T)$ such that $IR(S^i, d^i) = IR(T, d^i)$ for $i = 1, 2$. By INIR, we have $F(S^i, d^i) = F(T, d^i)$, which implies that $\delta(S^i) = \delta(T)$ for

$i = 1, 2$. Therefore, we have $\delta(S^1) = \delta(S^2)$.

Now let $S^1, S^2 \in \Gamma^2$ be such that $Int(PO(S^i) \neq \emptyset$ for $i = 1, 2$. Let $T \in \Gamma^2$ be such that $Int(PO(T)) \cap Int(PO(S^i)) \neq \emptyset$ for $i = 1, 2$. By applying the above argument twice, we have $\delta(S^i) = \delta(T)$ for $i = 1, 2$. Therefore, we conclude that $\delta(S^1) = \delta(S^2)$.

Q.E.D.

Theorem 7. *A solution satisfies PO, SY, INIR, d-CONT, P-CONT and R.D.CAV if and only if it is the lexicographic egalitarian solution.*

Proof. It is obvious that L satisfies the six axioms. Conversely, let F be a solution satisfying the six axioms. Lemma 6 implies that, for all $(S, d) \in \Sigma^2$ such that $F(S, d) \in Int(PO(S))$, $\ell(d, F(S, d))$ has the same slope. By SY, the slope should be equal to the 45 degree. Therefore, we establish that $F(S, d) = L(S, d)$ if $F(S, d) \in Int(PO(S))$. Given a problem $(S, d) \in \Sigma^2$, if $F(S, d) \notin Int(PO(S))$, then PO implies that it lies on the boundary of $PO(S)$. By Theorem 1, for all $x \in \ell(d, F(S, d)) \cap Int(S)$, $F(S, x) = F(S, d)$. By IR, we have $F(S, d) = L(S, d)$. Q.E.D.

5.3. Nash Solution

Now we discuss the most well-known solution in the axiomatic bargaining theory, the *Nash solution.* This solution, introduced and characterized by Nash [1950], has been extensively discussed in the literature. Properties which describe its behavior with respect to changes in the disagreement point have been investigated by Chun and Thomson [1987b], Peters [1986b] and Peters and van Damme [1987].

Definition. Given $\alpha \in Int(\Delta^1)$, the *weighted Nash solution with weights* α, N^α, is defined by setting, for all $(S, d) \in \Sigma^2$, $N^\alpha(S, d)$ to be the maximizer of the product

$\prod(x_i - d_i)^{\alpha_i}$ over $IR(S,d)$. The *Nash solution, N,* is the member of this family obtained by choosing $\alpha_1 = \alpha_2$.

To characterize the Nash solution, we introduce an invariance property. A *positive affine transformation* is a function $\lambda : \Re^2 \to \Re^2$ given by $a \in \Re^2_{++}$ and $b \in \Re^2$, such that for all $x \in \Re^2$, $\lambda(x) \equiv (a_1x_1 + b_1, a_2x_2 + b_2)$.

Scale Invariance (S.INV). For all $(S,d) \in \Sigma^2$ and for all positive affine transformations $\lambda : \Re^2 \to \Re^2$, $F(\lambda(S), \lambda(d)) = \lambda(F(S,d))$.

S.INV can be justified by the fact that agents' utility functions are von Neumann-Morgenstern types, which are unique up to positive affine transformations.

Nash [1950] showed that his solution is the unique solution satisfying PO, SY, S.INV and the following axiom.

Independence of Irrelevant Alternatives (IIA). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$, if $S^2 \subseteq S^1$, $d^2 = d^1$ and $F(S^1, d^1) \in S^2$, then $F(S^2, d^2) = F(S^1, d^1)$.

IIA requires that if an alternative has been judged superior to all others in some feasible set, then it should be judged superior to all others in any subset (to which it belongs) provided the disagreement point is kept constant.

Now we establish a characterization of the Nash solution by investigating the logical implication between IIA and our axioms.

Lemma 7. *Let F be a continuous linear solution. Then the solution satisfies IIA if and only if it satisfies INIR.*

Proof. It is clear that IR and IIA together imply INIR. To prove the converse statement, let $(S^1, d), (S^2, d) \in \Sigma^2$ be two problems such that $S^2 \subseteq S^1$ and $F(S^1, d) \in$

S^2. Now define the sequence of problems $\{(S^k, d^k)\}$ such that $S^2 \subseteq S^k \subseteq S^1$, $S^k \to S^2$, $d^k \in [d, F(S^1, d)[$ and $IR(S^k, d^k) = IR(S^1, d^k)$ for all k. By INIR, $F(S^k, d^k) = F(S^1, d^k)$ for all k. Since $d^k \in [d, F(S, d)[$ and F belongs to the linear family, $F(S^k, d^k) = F(S^k, d)$ and $F(S^1, d^k) = F(S^1, d)$ for all k. Altogether we have $F(S^k, d) = F(S^1, d)$ for all k. Since F is continuous, we conclude that $F(S^2, d) = F(S^1, d)$. Q.E.D.

Variants of the following theorem can be found in Chun and Thomson [1987b], Peters [1986b] and Peters and van Damme [1987]. Note that S-CONT and S.INV together imply d-CONT.

Theorem 8.[5] *A solution satisfies PO, INIR, S-CONT, D.Q-CAV and S.INV if and only if it is a weighted Nash solution.*

Remark 5. By dropping S-CONT from Theorem 8, the following solutions are permissible.

Definition. Given i, the i^{th} *benevolent dictatorial solution,* D^i, is defined by setting, for each $(S, d) \in \Sigma^2$, $D^i(S, d)$ equal to the point of $IR(S, d) \cap PO(S)$ preferred by agent i.

In fact, we can show that a solution satisfies PO, INIR, d-CONT, D.Q-CAV and S.INV if and only if it is a weighted Nash solution or a benevolent dictatorial solution. Since its proof is similar to that of Theorem 1 in Peters [1986b], we omit it.

Remark 6. D.Q-CAV in Theorem 8 can be replaced by the following axiom:

[5] This theorem can be generalized to n-person bargaining problems, as discussed in Chun and Thomson [1987b].

Restricted Disagreement Point Linearity (R.D.LIN). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$, $\alpha F(S, d^1) + (1-\alpha)F(S, d^2) \in PO(S)$ and S is smooth at both $F(S, d^1)$ and $F(S, d^2)$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) = \alpha F(S, d^1) + (1-\alpha)F(S, d^2).$$

For details, we refer to Chun and Thomson [1987b].

It is well-known that the Nash solution is the only weighted Nash solution satisfying the symmetry axiom.

Corollary 2. *A solution satisfies PO, INIR, d-CONT, D.Q-CAV, S.INV and SY if and only if it is the Nash solution.*

5.4. Kalai-Rosenthal Solution

Finally, we discuss the Kalai-Rosenthal [1978] solution.

Definition. The *Kalai-Rosenthal solution, KR,* is defined by setting, for all $(S, d) \in \Sigma^2$, $KR(S, d)$ be the maximal point of S on the segment connecting d and $a(S)$, where, for each i, $a_i(S) \equiv max\{x_i | x \in S\}$.

To characterize the Kalai-Rosenthal solution, we introduce two additional axioms. For all $(S, d) \in \Sigma^2$, let $T(S_d) \equiv comp\{(d_1, a_2(S)), (a_1(S), d_2)\}$.

Independence of Strongly Individually Rational Outcome (ISIR). For all $(S, d) \in \Sigma^2$ and for all $x \in \Re^2$, if $S = comp\{IR(S, d)\}$, $x \leq d$ and $F(T(S_d), x) = F(T(S_d), d)$, then $F(S, x) = F(S, d)$.

Strict Disagreement Point Monotonicity (S.D.MON). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all i, j such that $i \neq j$, if $S^1 = S^2$, $d_i^1 = d_i^2$, $d_j^1 < d_j^2$ and $a(S) \notin S$, then $F_j(S^2, d^2) > F_j(S^1, d^1)$.

ISIR, introduced by Peters [1986b], is interpreted as a weak form of path independence. S.D.MON, introduced by Livne [1987], requires that if an agent's utility at the disagreement point increases while the other's remains fixed, then the agent should gain strictly. This is a strengthening of the condition introduced by Thomson [1987].

A variant of the next theorem is in Peters [1986b].

Theorem 9. *A solution satisfies PO, IR, d-CONT, D.Q-CAV, SY, S.INV, ISIR and S.D.MON if and only if it is the Kalai-Rosenthal solution.*

Proof. It is clear that KR satisfies all eight axioms. Conversely, let F be a solution satisfying the eight axioms. From Theorem 2, it is a Pareto-optimal member of the linear solutions. Now by borrowing the proof of Theorem 4.1 in Peters [1986b], we can obtain the desired conclusion. Q.E.D.

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AXIOMS CONCERNING UNCERTAIN DISAGREEMENT POINTS FOR 2-PERSON BARGAINING PROBLEMS

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1. Introduction

Suppose bargaining takes place today, without the precise knowledge of the location of the disagreement point, this uncertainty being resolved tomorrow. Under what conditions will agents reach an agreement today? The minimal requirement is that each agent should be guaranteed at least the minimum of what he/she receives when the uncertainty is lifted tomorrow. Otherwise, the agent is definitely better off by waiting until tomorrow. We require that all agents should be guaranteed at least this minimum. This requirement of *disagreement point quasi-concavity* was introduced in Chun and Thomson [1987b] (variants of which are studied by Chun and Thomson [1987a], Livne [1986b], Peters [1986b] and Peters and van Damme [1987]). The purpose of this paper is to explore the implication of this axiom for 2-person bargaining problems.

To that purpose, we introduce a new family of solutions, which we call *linear solutions*. They are defined as follows. Let δ be a function associating with each problem a non-negative direction such that all interior points of the feasible set on the line passing through the disagreement point in the direction assigned by the function δ have the same direction. Then the *linear solution relative to* δ is defined by choosing as solution outcome of each problem the maximal feasible point such that the vector of utility gains from the disagreement point is in the direction determined by applying δ to the problem. This family of solutions, which we call the *linear family*, is fairly large, including many well-known solutions such as the Nash and egalitarian solutions. It also includes the lexicographic egalitarian and Kalai-Rosenthal solutions.

By imposing disagreement point quasi-concavity in conjunction with the standard conditions of weak Pareto optimality, individual rationality and continuity, we characterize continuous members of the linear family. Also, by strengthening weak Pareto optimality and weakening continuity, we characterize the Pareto optimal members of the linear family. Other characterizations of the family can be obtained by using axioms related to disagreement point quasi-concavity. We also show how well-known subfamilies or elements of the family can be singled out by imposing additional axioms.

The methodology, which we adopt here, is the axiomatic approach to bargaining theory, as introduced by Nash [1950]. However, the focus on the formulation of the bargaining problem is different. In the traditional formulation, it is typically assumed that the disagreement point is fixed. The possibility of varying disagreement points has recently been the object of a number of studies (Thomson [1987], Livne [1986a], Peters [1986b], and others). Moreover, bargaining situations in which the feasible set is known but the disagreement point is uncertain have been studied extensively (Chun and Thomson [1987a,b], Livne [1986b] and Peters and van Damme [1987]). The present paper is also focused on the role of uncertain disagreement points in bargaining.

The paper is organized as follows. Section 2 contains some preliminaries and introduces the basic axioms. Section 3 states our main axiom of disagreement point quasi-concavity, and characterizes the linear family. Section 4 discusses axioms related to disagreement point quasi-concavity and establishes alternative characterizations of the linear family. Finally, section 5 characterizes various subfamilies including the egalitarian, lexicographic egalitarian, Nash and Kalai-Rosenthal solutions.

2. Preliminaries

A 2-person bargaining problem, or simply a *problem,* is a pair (S, d), where S is a subset of $\Re^2$ and d is a point in S, such that

(1) S is convex and closed,

(2) $a_i(S) \equiv max\{x_i | x \equiv (x_1, x_2) \in S\}$ exists for $i = 1, 2$,

(3) S is *comprehensive*, i.e., for all $x \in S$ and for all $y \in \Re^2$, if $y \leq x$,[1] then $y \in S$,

(4) there exists $x \in S$ with $x > d$.

S is the *feasible set*. Each point x of S is a *feasible alternative*. The coordinates of x are the von Neumann-Morgenstern utility levels attained by the agents through the choice of some joint action. d is the *disagreement point* (or status quo). The intended interpretation of (S, d) is as follows: the agents can achieve any point of S if they unanimously agree on it. If they do not agree on any point, they end up at d. Let Σ^2 be the class of all problems and Γ^2 be the class of all feasible sets satisfying (1), (2) and (3).

A *solution* is a function $F: \Sigma^2 \to \Re^2$ such that for all $(S, d) \in \Sigma^2, F(S, d) \in S$. $F(S, d)$, the value taken by the solution F when applied to the problem (S, d), is called the *solution outcome of* (S, d).

The following axioms, which are standard in the literature, will be adopted whenever necessary.

Weak Pareto Optimality (WPO). For all $(S, d) \in \Sigma^2$ and for all $x \in \Re^2$, if $x > F(S, d)$, then $x \notin S$.

Pareto Optimality (PO). For all $(S, d) \in \Sigma^2$ and for all $x \in \Re^2$, if $x \geq F(S, d)$, then $x \notin S$.

Let $WPO(S) \equiv \{x \in S |$ for all $x' \in \Re^2, x' > x$ implies $x' \notin S\}$ be the set of *weakly Pareto optimal points of S*. Similarly, let $PO(S) \equiv \{x \in S |$ for all $x' \in \Re^2, x' \geq x$ implies $x' \notin S\}$ be the set of *Pareto optimal points of S*.

[1] Vector inequalities: given $x, y \in \Re^n, x \geqq y, x \geq y, x > y$.

Individual Rationality (IR). For all $(S,d) \in \Sigma^2$, $F(S,d) \geqq d$.

Let $IR(S,d) \equiv \{x \in S | x \geqq d\}$ be the set of *individually rational points of* (S,d).

d-Continuity (d-CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S,d) \in \Sigma^2$, if $S^k = S$ for all k and $d^k \to d$, then $F(S^k, d^k) \to F(S,d)$.

In the following, convergence of a sequence of sets is evaluated in the Hausdorff topology.

S-Continuity (S-CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S,d) \in \Sigma^2$, if $S^k \to S$ and $d^k = d$ for all k, then $F(S^k, d^k) \to F(S,d)$.

WPO requires that there be no feasible alternative at which all agents are better off than at the solution outcome. PO requires that the solution outcome should exhaust all gains from cooperation. IR requires that no agent be worse off at the solution outcome than at the disagreement point. Finally, d-CONT (respectively, S-CONT) requires that a small change in the disagreement point (respectively, the feasible set) cause only a small change in the solution outcome.

The following notation and terminology will be used frequently. Given $x_1, \ldots, x_k \in \Re^n$, $comp\{x_1, \ldots, x_k\}$ is the *comprehensive hull* of these points (the smallest comprehensive set containing them). Given $A \subset \Re^n$, $Int(A)$ is the relative interior of A. Δ^{n-1} is the (n-1)-dimensional simplex. Given $x \in \Re^2$ and $\delta \in \Delta^1$, $\ell(x,\delta)$ is the line passing through x in the direction δ. Finally, given $x, y \in \Re^2$ such that $x \neq y$, $\ell(x,y)$ is the line passing through x and y.

3. Disagreement Point Quasi-Concavity. The Main Characterization

The main purpose of this paper is to explore the implication of the following axiom, introduced by Chun and Thomson [1987b], for 2-person bargaining problems.

Disagreement Point Quasi-Concavity (D.Q-CAV). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$, for all i and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$, then

$$F_i(S, \alpha d^1 + (1-\alpha)d^2) \geqq min\{F_i(S, d^1), F_i(S, d^2)\}.$$

(Note that $(S, \alpha d^1 + (1-\alpha)d^2)$ is a well-defined element of Σ^2.)

This axiom can be motivated on the basis of timing of bargaining. Consider agents *today,* who, *tomorrow,* will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. The agents have two options: either they wait until tomorrow for the uncertainty to be lifted and solve then whatever problem has come up, or they consider the problem obtained by taking as disagreement point the average of d^1 and d^2 and solve that problem today. If, for some agent i, $F_i(S, \frac{d^1+d^2}{2})$ is smaller than the minimum of $F_i(S, d^1)$ and $F_i(S, d^2)$, then the agent will definitely prefer waiting until the uncertainty is lifted. For agent i to be persuaded that the problem should be solved today, he should be guaranteed at least the minimum of $F_i(S, d^1)$ and $F_i(S, d^2)$. Imposing D.Q-CAV provides this minimum incentive to all agents.

We are interested in the following new family of solutions, which generalizes the egalitarian, lexicographic egalitarian, Nash and Kalai-Rosenthal solutions.

Definition. Let $\delta : \Sigma^2 \rightarrow \Delta^1$ be a function such that for all $S \in \Gamma^2$ and for all $y \in Int(S)$, $y \in \ell(d, \delta(S, d))$ implies that $\delta(S, y) = \delta(S, d)$. The *linear solution relative to* δ, F^δ, is defined by setting, for each $(S, d) \in \Sigma^2$, $F^\delta(S, d)$ equal to $\ell(d, \delta(S, d)) \cap WPO(S)$.

Note that, for the solution F^δ to be well-defined, it should be that for all $S \in \Gamma^2$ (i) for all $d^1, d^2 \in Int(S)$, if $\delta(S,d^1) \neq \delta(S,d^2)$, then $\ell(d^1,\delta(S,d^1)) \cap \ell(d^2,\delta(S,d^2)) \cap Int(S) = \emptyset$, and (ii) $\delta(S,\cdot)$ is continuous with respect to d.

We now turn to the results. The proof of Lemma 1 is the same as the proof of Lemma 1 in Chun and Thomson [1987b].

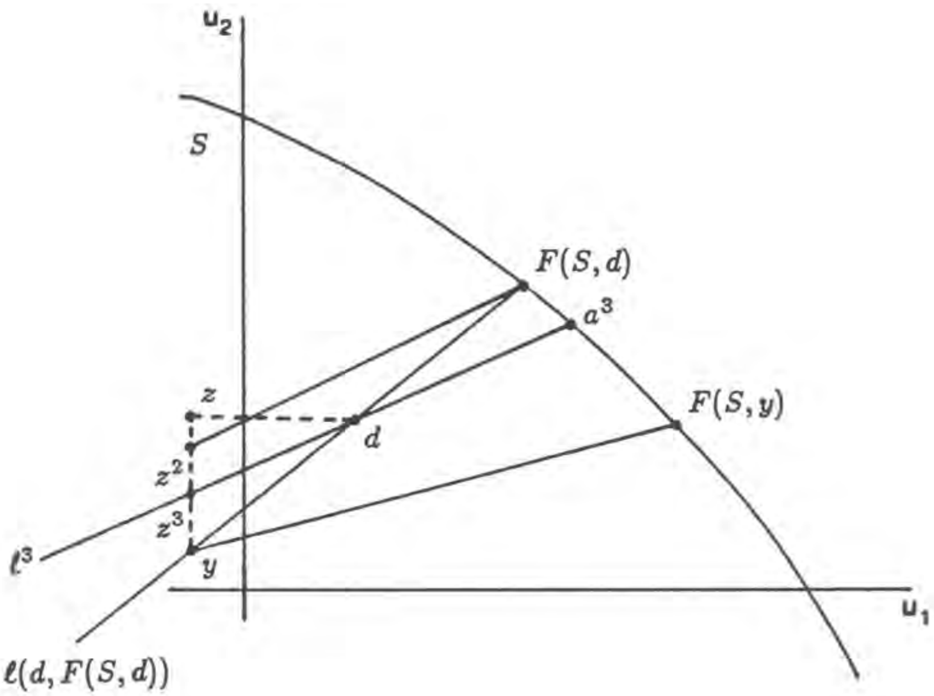
Lemma 1. *Let F be a solution satisfying WPO, IR and D.Q-CAV. Also let $(S,d) \in \Sigma^2$ be such that $F(S,d) \in PO(S)$. Then for all $x \in [d, F(S,d)[$, $F(S,x) = F(S,d)$.*

Proof. First, note that $(S,x) \in \Sigma^2$ for all $x \in [d, F(S,d)[$. Let $x \in]d, F(S,d)[$ be given. Let $\bar{\lambda} \in]0,1[$ be such that $x = \bar{\lambda} d + (1-\bar{\lambda})F(S,d)$, and $\{\lambda^k\} \subset]0,1[$ be such that $\lambda^k < \bar{\lambda}$ for all k and $\lambda^k \to \bar{\lambda}$. Also, let $x^k \equiv \frac{x-\lambda^k d}{1-\lambda^k}$ for all k. Note that $(S,x^k) \in \Sigma^2$ for all k. By D.Q-CAV, $F_i(S,x) \geqq min\{F_i(S,x^k), F_i(S,d)\}$ for all i and for all k. As $k \to \infty$, $x^k \to F(S,d)$ and since $F(S,d) \in PO(S)$, it follows from IR that $F(S,x^k) \to F(S,d)$. Therefore, we obtain $F(S,x) \geqq F(S,d)$. Since $F(S,d) \in PO(S)$, we conclude that $F(S,x) = F(S,d)$. Q.E.D.

Lemma 2. *Let F be a solution satisfying WPO, IR, d-CONT and D.Q-CAV. Also, let $(S,d) \in \Sigma^2$ be such that $F(S,d) \in Int(PO(S))$. Then for all $x \in \ell(d, F(S,d)) \cap Int(S)$, $F(S,x) = F(S,d)$.*

Proof. Let F and $(S,d) \in \Sigma^2$ satisfying the hypothesis of Lemma 2 be given. From Lemma 1, we know that for all $x \in [d, F(S,d)[$, $F(S,x) = F(S,d)$. Now suppose, by way of contradiction, that there exists $y \in Int(S)$ such that $d \in]y, F(S,d)[$ and $F(S,y) \neq F(S,d)$. Since $F(S,d) \in Int(PO(S))$, it follows from WPO and d-CONT, we can assume that $F(S,y) \in Int(PO(S))$.

(a) We consider the case when $\ell(d, F(S,d))$ is neither horizontal nor vertical. Suppose that $F_1(S,y) > F_1(S,d)$. Let $z \equiv (y_1, d_2)$.



Proof of Lemma 2.
Figure 1.

Claim 1. $F_1(S,z) \le F_1(S,d)$.

Otherwise, from WPO and d-CONT, there exists $z^1 \in \,]z,d[$ such that $F_1(S,d) < F_1(S,z^1) \le F_1(S,z)$ and $F(S,z^1) \in PO(S)$. From Lemma 1, for all $x \in [z^1, F(S,z^1)[$, $F(S,x) = F(S,z^1)$. Since $F_2(S,z^1) \ge z_2^1 = d_2$ by IR, there exists an $\bar{x} \in [d, F(S,d)[\cap [z^1, F(S,z^1)[$, which is a contradiction.

Claim 2. $F_1(S,y) > F_1(S,d)$ is impossible.

Since $F_1(S,z) \leq F_1(S,d)$, by d-CONT, there exists $z^2 \in [z, y[$ such that $F(S,z^2) = F(S,d)$. From Lemma 1, for all $x \in [z^2, F(S,d)[$, $F(S,x) = F(S,d)$. Also, from WPO, IR and Lemma 1, we have for all $x \in [z^2, d]$, $F(S,x) = F(S,d)$. Now define the sequence of problems $\{(S,z^k)\}$ by $z^{k+1} \equiv \frac{1}{2}(z^k + y)$ for all $k \geq 2$. Also, for all $k \geq 3$, let ℓ^k be the line passing through z^k and d, and $a^k \equiv \ell^k \cap WPO(S)$. For all $x \in]z^2, z^3]$, if $F_1(S,d) < F_1(S,x) \leq min\{F_1(S,y), a_1^3\}$, then there exists z' such that $z' \in \ell(x, F(S,x)) \cap \ell(z^2, d)$. Since we assumed that $F(S,x) \neq F(S,d)$, this is impossible. Therefore, for all $x \in]z^2, z^3]$, we have $F_1(S,x) \leq F_1(S,d)$ or $F_1(S,x) > min\{F_1(S,y), a_1^3\}$. By d-CONT, we have $F_1(S,x) \leq F_1(S,d)$ for all $x \in [z^2, z^3]$. By repeating the same procedure, for all $x \in [z^2, y[$, we obtain $F_1(S,x) \leq F_1(S,d)$. Therefore, $F_1(S,y) > F_1(S,d)$ contradicts d-CONT.

By a similar argument, we obtain a contradiction to $F_1(S,y) < F_1(S,d)$.

(b) Now suppose that $\ell(d, F(S,d))$ is horizontal and there exists $y \in Int(S)$ such that $d \in]y, F(S,d[$ and $F(S,y) \neq F(S,d)$. By IR and d-CONT, there exists $z^1 \in [y, d[$ such that $F(S,z^1) \neq F(S,d)$ and that $\ell(z^1, F(S,z^1))$ is positively sloped. From (a), for all $z \cap \ell(z^1, F(S,z^1)) \cap Int(S)$, $F(S,z) = F(S,z^1)$. Now let a^* be the Pareto optimal point of S on the line passing through d parallel to $\ell(z^1, F(S,z^1))$. For some $z \in [z^1, d[$, say z^2, if $\ell(z^2, F(S,z^2))$ is flatter than $\ell(z^1, F(S,z^1))$, then there exists $z' \in Int(S)$ such that $z' \in \ell(z^1, F(S,z^1)) \cap \ell(z^2, F(S,z^2))$, which is impossible. Therefore, for all $z \in [z^1, d[$, $F_1(S,z) < a_1^*$. This is incompatible with d-CONT. A similar argument can be established when $\ell(d, F(S,d))$ is vertical. Q.E.D.

Remark 1. Lemma 1 can easily be generalized to n-person problems. However, it remains an open question whether Lemma 2 can be generalized to such problems.

Now we present our main results.

Theorem 1. *A solution satisfies PO, IR, d-CONT and D.Q-CAV if and only if it is a linear solution F^δ with the additional property, that for all $(S,d) \in \Sigma^2$, $\ell(d,\delta(S,d)) \cap WPO(S) \backslash PO(S) = \emptyset$.*

Proof. It is obvious that all F^δ satisfy IR, d-CONT and D.Q-CAV, and if δ satisfies the additional property, PO. Conversely, let F be a solution satisfying the four axioms. For all $(S,d) \in \Sigma^2$, let $\delta(S,d) \equiv \frac{F(S,d)-d}{||F(S,d)-d||}$. Since PO and IR together imply that $F(S,d) \geq d$, δ is a well-defined function from Σ^2 to Δ^1. It is enough to show that for all $(S^1,d^1),(S^2,d^2) \in \Sigma^2$, if $S^1 = S^2 \equiv S$ and $d^2 \in \ell(d^1,F(S,d^1))$, then $\delta(S,d^2) = \delta(S,d^1)$. If $F(S,d^1) \in Int(PO(S))$, then the desired conclusion follows from Lemma 2. Suppose now that $F(S,d^1) \notin Int(PO(S))$ and that $\delta(S,d^1) \neq \delta(S,d^2)$. From Lemma 1, for all $d \in [d^1,F(S,d^1)[$, $F(S,d) = F(S,d^1)$ and for all $d \in [d^2,F(S,d^2)[$, $F(S,d) = F(S,d^2)$. By PO and d-CONT, there exists $d' \in]d^1,d^2[$ such that $F(S,d') \in Int(PO(S))$, $F(S,d') \neq F(S,d^2)$ and that either $\ell(d',F(S,d')) \cap [d^1,F(S,d^1)[\neq \emptyset$ or $\ell(d',F(S,d')) \cap [d^2,F(S,d^2)[\neq \emptyset$. Since $F(S,d') \neq F(S,d^1)$ and $F(S,d') \neq F(S,d^2)$, it is a contradiction.

Finally, we note that PO implies that, for all $(S,d) \in \Sigma^2$, $\ell(d,\delta(S,d)) \cap WPO(S) \backslash PO(S) = \emptyset$. Q.E.D.

Remark 2. The family of solutions characterized in Theorem 1 is fairly large, including the Nash, Kalai-Rosenthal and lexicographic egalitarian solutions. However, the egalitarian solution is excluded, since it violates PO.

Theorem 2. *A solution satisfies WPO, IR, d-CONT, S-CONT and D.Q-CAV if and only if it is a linear solution F^δ with the additional property, that $\delta(\cdot, x)$ be continuous with respect to S.*

Proof. It is obvious that all F^δ satisfy WPO, IR, d-CONT and D.Q-CAV, and if $\delta(\cdot, x)$ is continuous with respect to S, S-CONT. Conversely, let F be a solution satisfying the five axioms. For all $(S, d) \in \Sigma^2$, let $\delta(S, d) \equiv \frac{F(S,d)-d}{||F(S,d)-d||}$. Since WPO and IR together imply that $F(S, d) \geq d$, δ is a well-defined function from Σ^2 to Δ^1. It is enough to show that, for all $(S, d) \in \Sigma^2$, if there exists $d' \in \ell(d, F(S, d)) \cap Int(S)$, then $\delta(S, d') = \delta(S, d)$. If $F(S, d) \in Int(PO(S))$, then the desired conclusion follows from Lemma 2. Otherwise, let $\{(S^k, d)\} \subset \Sigma^2$ be a sequence of problems such that for all k, $F(S^k, d) \in Int(PO(S^k))$ and $d \in Int(S^k)$ and such that $S^k \to S$. By the previous argument, $F(S^k, d) = F^\delta(S^k, d)$ for all k, and by S-CONT, $F(S, d) = F^\delta(S, d)$.

Finally, we note that S-CONT implies the continuity of $\delta(\cdot, x)$ with respect to S in the Hausdorff topology. Q.E.D.

Remark 3. The family of solutions characterized in Theorem 2 is fairly large, including the Nash, egalitarian and Kalai-Rosenthal solutions. However, the lexicographic egalitarian solution is excluded, since it violates S-CONT.

4. Variants of the Main Result

Recently, bargaining situations in which the feasible set is known but the disagreement point is uncertain have been studied extensively. Several axioms related to disagreement point quasi-concavity have appeared. Here we discuss how the linear family can be characterized using these axioms.

The first axiom, which we call *weak disagreement point linearity,* [2] was introduced by Livne [1986b] in his study of the Nash solution.

Weak Disagreement Point Linearity (W.D.LIN). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^n$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$ and $F(S, d^1) = F(S, d^2) \equiv x$, then $F(S, \alpha d^1 + (1-\alpha)d^2) = x$.

Again, this axiom can be motivated on the basis of timing of bargaining. Suppose agents *today,* who, *tomorrow,* will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. Suppose that the solution outcomes of the two problems are coincide. Since all agents receive the same amount tomorrow irrespective of the uncertainty, it is natural to require that they should receive the same amount today. Imposing W.D.LIN on the solutions makes the uncertainty not affect the final allocation.

Now we explore the implication of this axiom for 2-person bargaining problems. By replacing D.Q-CAV by W.D.LIN in Theorems 1 and 2, we obtain the same conclusions. In addition, by using the following weak condition, a characterization of the linear family can be established.

Boundary (BOUND). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S, d) \in \Sigma^2$, if $S^k = S$ for all k, $F(S, d) = x$ and $d^k \to x$, then $F(S^k, d^k) \to x$.

For a solution satisfying Pareto optimality, BOUND is just a considerable weakening of IR. For a solution satisfying only weak Pareto optimality, BOUND is a weak continuity property requiring that if the disagreement point is closer to the boundary of the

[2] He calls this property 'Independence of Convex Combination of Equivalent Conflict Outcomes.'

feasible set, then the solution outcome is also closer to the disagreement point. It is a very weak condition satisfied by all well-known solutions.

Now we have the following result.

Lemma 3. *Let F be a solution satisfying WPO, IR, d-CONT, BOUND and W.D.LIN. Also, let $(S,d) \in \Sigma^2$ be given. Then for all $x \in [d, F(S,d)[$, $F(S,x) = F(S,d)$.*

Proof. First, note that for all $x \in [d, F(S,d)[$, $(S,x) \in \Sigma^2$. We assume that $WPO(S)$ contains a vertical segment. The case that $WPO(S)$ contains a horizontal (or both vertical and horizontal) segment can be dealt with similarly. Now suppose, by way of contradiction, that there exists $d^1 \in [d, F(S,d)[$ such that $F(S,d^1) \neq F(S,d)$. Two cases are possible:

(i) $F_2(S,d^1) > F_2(S,d)$.

Note that if $F_2(S,d^1) > F_2(S,d)$, IR implies that $\ell(d, F(S,d))$ is not vertical. Let $d^2 \in Int(S)$ be such that $d_1^2 = d_1$ and that for all $a \in IR(S,d^1)$, $a_2 > F_2(S,d^1)$. By WPO, $F(S,d^2) \in WPO(S)$ and by IR, $F_2(S,d^2) > F_2(S,d^1)$. By d-CONT, there exists $d^3 \in [d^2, d[$ such that $F(S,d^3) = F(S,d^1)$. By W.D.LIN, for all $d' \in [d^1, d^3]$, $F(S,d') = F(S,d^1)$.

Now let $d(\lambda)$ be a parametrization of $[d^1, F(S,d)]$ such that $d(0) = d^1$ and $d(1) = F(S,d)$. By d-CONT, $F(S,d(\lambda))$ moves continuously. By BOUND, there exists $\bar{\lambda} \in [0,1[$ such that $F_2(S,d^1) > F_2(S,d(\bar{\lambda})) \geqq F_2(S,d)$. Let $d(\bar{\lambda}) = d^4$. Also, by d-CONT, there exists $d^5 \in [d^3, d]$ such that $F(S,d^5) = F(S,d^4)$. By W.D.LIN, for all $d' \in [d^4, d^5]$, $F(S,d') = F(S,d^4)$. Then $[d^1, d^3]$ and $[d^4, d^5]$ intersect. Let d^6 be the intersection point. Clearly, $d^6 \in Int(S)$. Since $F(S,d^1) \neq F(S,d^4)$, it is a contradiction.

(ii) $F_2(S, d^1) < F_2(S, d)$.

From the same argument as in (i), we have, for all $d' \in [d^1, F(S, d^1)[$, $F_2(S, d') \leqq F_2(S, d^1)$. Let d^2 be a point in $]d^1, F(S, d^1)[$.

Let $d(\lambda)$ be a parametrization of $[d^2, F(S, d)]$ such that $d(0) = d^2$ and $d(1) = F(S, d)$. By d-CONT, $F(S, d(\lambda))$ moves continuously. By BOUND, there exists $\bar{\lambda} \in [0, 1[$ such that $F_2(S, d) \geqq F_2(S, d(\bar{\lambda})) > F_2(S, d^1)$. Let $d(\bar{\lambda}) = d^3$. Also, by d-CONT, there exists $d^4 \in [d^2, d]$ such that $F(S, d^4) = F(S, d^3)$. By W.D.LIN, for all $d' \in [d^3, d^4]$, $F(S, d') = F(S, d^3)$. Then $[d^1, F(S, d^1)[$ and $[d^3, d^4]$ intersect. Let d^5 be the intersection point. Clearly, $d^5 \in Int(S)$. Since $d^5 \in [d^1, F(S, d^1)[$, $F_2(S, d^5) \leqq F_2(S, d^1)$, and since $d^5 \in [d^3, d^4]$, $F_2(S, d^5) = F_2(S, d^3) > F_2(S, d^1)$. This is a contradiction. Q.E.D.

Theorem 3. *A solution satisfies WPO, IR, d-CONT, BOUND and W.D.LIN if and only if it is a linear solution.*

Proof. It is obvious that all F^δ satisfy the five axioms. Conversely, let F be a solution satisfying the five axioms. First, we know from Lemma 3 that for all $(S, d) \in \Sigma^2$, and for all $x \in [d, F(S, d)[$, $F(S, x) = F(S, d)$. Now we extend the conclusion of Lemma 3 to all $x \in \ell(d, F(S, d)) \cap Int(S)$. Since the proof is similar to that of Lemma 2, we omit it. Q.E.D.

The second axiom was introduced by Peters and van Damme [1987] [3] in their study of the Nash solution.

[3] They call this property 'convexity.'

Disagreement Point Linearity (D.LIN). For all $(S,d) \in \Sigma^2$ and for all $\alpha \in]0,1]$, $F(S, \alpha d + (1-\alpha)F(S,d)) = F(S,d)$.

This is a strengthening of W.D.LIN to require that, for a given problem (S,d), a new problem obtained by taking the same feasible set and a different disagreement point, which is a convex combination of the old disagreement point and its solution outcome, should yield the same solution outcome. If we extend our domain of bargaining problems to allow the disagreement point to lie on the boundary of the feasible set, and define the solution outcome of such problems be the disagreement point, then the motivation similar to W.D.LIN can be given.

Now we explore the implication of this axiom for 2-person bargaining problems. Again, by replacing D.Q-CAV by D.LIN in Theorems 1 and 2, we obtain the same conclusion. In addition, the following theorem can be established.

Theorem 4. *A solution satisfies WPO, IR, d-CONT and D.LIN if and only if it is a linear solution.*

Proof. It is obvious that all F^δ satisfy the four axioms. The converse statement is established by exploiting the logical implications between D.LIN, W.D.LIN and BOUND. Indeed, it can easily be shown that (i) WPO and D.LIN together imply W.D.LIN, and that (ii) d-CONT and D.LIN together imply BOUND. Therefore, by Theorem 3, we obtain the desired conclusion. Q.E.D.

Remark 4. If IR in the Theorem 4 is dropped from the list, then following family of the *generalized linear solutions* can be characterized. Let $B^1 \equiv \{x \in \Re^2 | \sum |x_i| = 1$ and $-x \notin \Re^2_+\}$ and given $x \in \Re^2$ and $\delta \in B^1$, let $\bar{\ell}(d,\delta)$ be the line passing through d

in the direction δ. Also, given $(S,d) \in \Sigma^2$, let $\bar{\ell}(d,\delta) \cap WPO(S)$ be the weakly Pareto optimal point of S on the half-line passing through d in the direction δ.

Definition. Let δ be a function such that, for all $(S,d) \in \Sigma^2$, $\delta(S,d) \in B^1$ and that, for all $S \in \Gamma^2$ and for all $y \in Int(S)$, $y \in \bar{\ell}(d,\delta(S,d))$ implies that $\delta(S,y) = \delta(S,d)$ and that $\delta(S,\cdot)$ is continuous with respect to d. Given the function δ, the *generalized linear solution relative to* δ is defined by setting, for each $(S,d) \in \Sigma^2$, $F^\delta(S,d)$ equal to $\bar{\ell}(d,\delta(S,d)) \cap WPO(S)$.

5. Further Characterizations

In this section, we impose additional axioms (or strengthen the axioms used in the Theorems 1 and 2) to characterize important subfamilies of the linear family.

5.1. Egalitarian Solution

First, we consider a subfamily of the linear solutions, which generalizes the well-known egalitarian solution (Kalai [1977] and Thomson and Myerson [1980]).

Definition. Given a continuous function $\delta : \Gamma^2 \to \Delta^1$, the *directional solution relative to* δ, E^δ, is defined by setting, for all $(S,d) \in \Sigma^2$, $E^\delta(S,d)$ equal to $\ell(d,\delta(S)) \cap WPO(S)$. Given $\alpha \in \Delta^1$, the *weighted egalitarian solution with weights* α, E^α, is defined by setting, for all $(S,d) \in \Sigma^2$, $E^\alpha(S,d)$ equal to $\ell(d,\alpha) \cap WPO(S)$. The *egaltarian solution* is obtained by choosing $\alpha_1 = \alpha_2$.

This family can be characterized by the following axiom, which strengthens D.Q-CAV.

Disagreement Point Concavity (D.CAV). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) \geqq \alpha F(S, d^1) + (1-\alpha)F(S, d^2).$$

This axiom, introduced and studied in Chun and Thomson [1987a], gives an even greater incentive to all agents to reach an agreement today than D.Q-CAV does. Consider agents *today*, who, *tomorrow*, will face one of two equally likely problems (S, d^1) and (S, d^2), having the same feasible set, but different disagreement points. The agents have two options: either they wait until tomorrow for the uncertainty to be lifted and solve then whatever problem has come up, or they consider the problem obtained by taking as disagreement point the average of d^1 and d^2 and solve that problem today. The expected payoff associated with the first option is $\frac{F(S,d^1)+F(S,d^2)}{2}$ and that associated with the second option is $F(S, \frac{d^1+d^2}{2})$, since $\frac{d^1+d^2}{2}$ is the corresponding "expected" disagreement point. If either $F(S, \frac{d^1+d^2}{2})$ weakly dominates $\frac{F(S,d^1)+F(S,d^2)}{2}$ or the reverse holds, all agents agree on when to do. A conflict may arise if neither of these inequalities holds. Imposing D.CAV on the solutions prevents such a conflict.

It can easily be checked that D.CAV implies D.Q-CAV. D.CAV can be regarded as a dual to an axiom considered by Myerson [1981] concerning uncertainty in the feasible set (variants of which are studied by Perles and Maschler [1981] and Peters [1986a]).

The following results, which can be generalized to n-person bargaining problems, are due to Chun and Thomson [1987a]. We note that d-CONT is not used.

Theorem 5. *A solution satisfies WPO, IR, S-CONT and D.CAV if and only if it is a directional solution.*

The family of weighted egalitarian solution can be characterized by strengthening IR to the following axiom.

Independence of Non-Individually Rational Alternatives (INIR). For all $(S,d) \in \Sigma^2$, $F(S,d) = F(comp\{IR(S,d)\}, d)$.

This axiom, introduced by Peters [1986a], says that the non-individually rational alternatives are irrelevant to the determination of the solution outcome. It is a natural condition since agents are guaranteed their utilities at d. It can easily be checked that WPO, INIR and S-CONT (or PO and INIR) together imply IR.

Theorem 6. *A solution satisfies WPO, INIR, S-CONT and D.CAV if and only if it is a weighted egalitarian solution.*

The egalitarian solution is the only weighted egalitarian solution satisfying the following axiom.

Symmetry (SY). For all $(S,d) \in \Sigma^2$ and for all permutations $\pi : \{1,2\} \to \{1,2\}$, if $S = \pi(S)$ and $d = \pi(d)$, then $F_1(S,d) = F_2(S,d)$.

Sy says that if the only information available on the conflict situation is contained in the mathematical description of (S,d), and (S,d) is a symmetric problem, then there is no ground for favoring one agent at the expense of another.

Corollary 1. *A solution satisfies WPO, INIR, S-CONT, D.CAV and SY if and only if it is the egalitarian solution.*

5.2. Lexicographic Egalitarian Solution

The directional solutions often violate PO. The following extension, called the *lexicographic egalitarian solution*, is an adaptation of the egalitarian solution that satisfies PO. However, we note that this solution does not satisfy S-CONT.

Definition. The *lexicographic egalitarian solution, L,* is defined by setting, for all $(S,d) \in \Sigma^2$, $L(S,d) = E(S,d)$ if $E(S,d) = PO(S)$ and $L(S,d) = \{x \in PO(S) | x_1 = E_1(S,d) \text{ or } x_2 = E_2(S,d)\}$, otherwise. [4]

Chun and Thomson [1987a] showed that there is no solution satisfying PO, IR and D.CAV. However, the following slight weakening of D.CAV is compatible with PO and IR.

Restricted Disagreement Point Concavity (R.D.CAV). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$ and $F(S,d^1), F(S,d^2) \in Int(PO(S))$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) \geqq \alpha F(S,d^1) + (1-\alpha)F(S,d^2).$$

The motivation for this axiom is same as for D.CAV, except that the conclusion is required to hold for the interior of the Pareto optimal set. For a solution satisfying PO, if it chooses the boundary point of the Pareto optimal set as the solution outcome, then the solution outcome becomes less sensitive to changes in the disagreement point. Therefore, it is unreasonable to require that the solution behave well even on the boundary. It can easily be checked that PO, d-CONT and R.D.CAV (or PO, IR and R.D.CAV) together imply D.Q-CAV.

To characterize the lexicographic egalitarian solution, S-CONT is weakened to the following condition.

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[4] This solution has been studied by Imai [1983] and Lensberg [1982].

Pareto-Continuity (P-CONT). For all sequences $\{(S^k, d^k)\} \subset \Sigma^2$ and for all $(S, d) \in \Sigma^2$, if $S^k \to S$, $PO(S^k) \to PO(S)$ and $d^k = d$ for all k, then $F(S^k, d^k) \to F(S, d)$.

P-CONT requires that a small change in the feasible set and the Pareto optimal set causes only a small change in the solution outcome. It can easily be checked that S-CONT implies P-CONT.

Now we characterize the lexicographic egalitarian solution. The proof is similar to that of Theorems 1 and 6 of Chun and Thomson [1987a], which characterize the egalitarian solution. We note again that PO and INIR together imply IR.

Lemma 4. *Let F be a solution satisfying PO, IR, d-CONT and R.D.CAV. Also let $S \in \Gamma^2$ and $d^1, d^2 \in Int(S)$. If $\alpha F(S, d^1) + (1-\alpha)F(S, d^2) \in Int(PO(S))$ for all $\alpha \in [0, 1]$, then the line $\ell(d^1, F(S, d^1))$ is parallel to the line $\ell(d^2, F(S, d^2))$.*

Proof. Let S, d^1 and d^2 be as in the Lemma. By Theorem 1, for all $y \in \ell(d^1, F(S, d^1)) \cap Int(S)$, $F(S, y) = F(S, d)$. Let $d^3 \in \ell(d^1, F(S, d^1)) \cap Int(S)$ be such that $d^3 \neq d^1$. Without loss of generality, suppose that $d^3 \in \,]d^1, F(S, d^1)[$. Now let $z^i \equiv \frac{1}{2}(d^i + d^2)$ for $i = 1, 3$. By R.D.CAV, we have

$$F(S, z^i) \geqq \frac{1}{2}\{F(S, d^i) + F(S, d^2)\} = \frac{1}{2}\{F(S, d^1) + F(S, d^2)\} \equiv x \quad \text{for} \quad i = 1, 3.$$

Since $x \in PO(S)$, we have $F(S, z^i) = x$ for $i = 1, 3$. For all $z \in \ell(z^1, z^2) \cap Int(S)$ such that $z^3 = \lambda z^1 + (1-\lambda)z$ for some $\lambda \in \,]0, 1[$, by R.D.CAV and $\lambda < 1$, we have $x \geqq F(S, z)$. Since $x \in Int(PO(S))$, we have, by PO, $F(S, z) = x$. By IR, $\ell(z^1, z^2)$ passes through x. This is possible only if $\ell(d^1, F(S, d^1))$ is parallel to $\ell(d^2, F(S, d^2))$.

Q.E.D.

Lemma 5. *Let F be a solution satisfying PO, IR, d-CONT, P-CONT and R.D.CAV.*

Also let $S \in \Gamma^2$ be given. Then for all $(S,d) \in \Sigma^2$ such that $F(S,d) \in Int(PO(S))$, $\ell(d, F(S,d))$ has the same slope.

Proof. Let $S \in \Gamma^2$ be a polygonal feasible set such that $Int(PO(S)) \neq \emptyset$. Let $\{S^i | i \in I\}$, where $I \subseteq N$ and $S^i \equiv \{x \in \Re^2 | \sum p^i_j x_j \leqq c^i$ for some $p^i \in \Delta^1$ and $c^i \in \Re\}$ be a minimal collection such that $S = \cap_{i \in I} S^i$. Let $i \in I$ be such that $p^i > 0$. By PO and IR, there exists $x \in Int(S)$ such that $F(S,x) \in PO(S^i) \cap Int(PO(S))$. By Lemma 4, for all $d \in Int(S)$, if $F(S,d) \in PO(S^i) \cap Int(PO(S))$, then the line $\ell(x, F(S,x))$ is parallel to the line $\ell(d, F(S,d))$. Let the common direction be denoted by $\delta(S^i)$. By IR, $\delta(S^i) \geq 0$. Also, for all $d \in Int(S)$, if $\ell(d, \delta(S^i)) \cap PO(S^i) \equiv y \in Int(PO(S))$, $F(S,d) = y$ from Lemma 4.

Now let S^i and S^j be such that $PO(S^i) \cap PO(S^j) \neq \emptyset$. Note that $a \in PO(S^i) \cap PO(S^j)$ implies that $a \in Int(PO(S))$. Without loss of generality, suppose that $i = 1$ and $j = 2$. We claim that $\delta(S^1) = \delta(S^2)$. Let $a \in PO(S^1) \cap PO(S^2)$, $y^1 \in PO(S^1) \cap Int(PO(S))$, $d^1 \equiv y^1 - \delta(S^1)$ and $d^2 \equiv a - \delta(S^2)$. By the previous step, $F(S,d^1) = y^1$ and $F(S,d^2) = a$. By Lemma 4 applied to d^1 and d^2, we conclude that $\ell(d^1, F(S,d^1))$ is parallel to $\ell(d^2, F(S^2)) = \ell(d^2, a)$. Therefore, we have $\delta(S^1) = \delta(S^2)$. Repeating the argument, we have $\delta(S^i) = \delta(S^1)$ for all S^i with $p^i > 0$, as desired. Q.E.D.

Lemma 6. *Let F be a solution satisfying PO, INIR, d-CONT, P-CONT and R.D.CAV. Then for all $(S,d) \in \Sigma^2$ such that $F(S,d) \in Int(PO(S))$, $\ell(d, F(S,d))$ has the same slope.*

Proof. Let $S^1, S^2 \in \Gamma^2$ be such that $Int(PO(S^1)) \cap Int(PO(S^2)) \neq \emptyset$. For $i = 1,2$, let $\delta(S^i)$ be the direction derived in the proof of Lemma 5. Also, let $T \equiv S^1 \cap S^2$. Since $Int(PO(S^i)) \neq \emptyset$, there exists $d^i \in Int(T)$ such that $IR(S^i, d^i) = IR(T, d^i)$ for $i = 1,2$. By INIR, we have $F(S^i, d^i) = F(T, d^i)$, which implies that $\delta(S^i) = \delta(T)$ for

$i = 1, 2$. Therefore, we have $\delta(S^1) = \delta(S^2)$.

Now let $S^1, S^2 \in \Gamma^2$ be such that $Int(PO(S^i) \neq \emptyset$ for $i = 1, 2$. Let $T \in \Gamma^2$ be such that $Int(PO(T)) \cap Int(PO(S^i)) \neq \emptyset$ for $i = 1, 2$. By applying the above argument twice, we have $\delta(S^i) = \delta(T)$ for $i = 1, 2$. Therefore, we conclude that $\delta(S^1) = \delta(S^2)$.

Q.E.D.

Theorem 7. *A solution satisfies PO, SY, INIR, d-CONT, P-CONT and R.D.CAV if and only if it is the lexicographic egalitarian solution.*

Proof. It is obvious that L satisfies the six axioms. Conversely, let F be a solution satisfying the six axioms. Lemma 6 implies that, for all $(S, d) \in \Sigma^2$ such that $F(S, d) \in Int(PO(S))$, $\ell(d, F(S, d))$ has the same slope. By SY, the slope should be equal to the 45 degree. Therefore, we establish that $F(S, d) = L(S, d)$ if $F(S, d) \in Int(PO(S))$. Given a problem $(S, d) \in \Sigma^2$, if $F(S, d) \notin Int(PO(S))$, then PO implies that it lies on the boundary of $PO(S)$. By Theorem 1, for all $x \in \ell(d, F(S, d)) \cap Int(S)$, $F(S, x) = F(S, d)$. By IR, we have $F(S, d) = L(S, d)$. Q.E.D.

5.3. Nash Solution

Now we discuss the most well-known solution in the axiomatic bargaining theory, the *Nash solution*. This solution, introduced and characterized by Nash [1950], has been extensively discussed in the literature. Properties which describe its behavior with respect to changes in the disagreement point have been investigated by Chun and Thomson [1987b], Peters [1986b] and Peters and van Damme [1987].

Definition. Given $\alpha \in Int(\Delta^1)$, the *weighted Nash solution with weights* α, N^α, is defined by setting, for all $(S, d) \in \Sigma^2$, $N^\alpha(S, d)$ to be the maximizer of the product

$\prod(x_i - d_i)^{\alpha_i}$ over $IR(S, d)$. The *Nash solution, N*, is the member of this family obtained by choosing $\alpha_1 = \alpha_2$.

To characterize the Nash solution, we introduce an invariance property. A *positive affine transformation* is a function $\lambda : \Re^2 \to \Re^2$ given by $a \in \Re^2_{++}$ and $b \in \Re^2$, such that for all $x \in \Re^2$, $\lambda(x) \equiv (a_1 x_1 + b_1, a_2 x_2 + b_2)$.

Scale Invariance (S.INV). For all $(S, d) \in \Sigma^2$ and for all positive affine transformations $\lambda : \Re^2 \to \Re^2$, $F(\lambda(S), \lambda(d)) = \lambda(F(S, d))$.

S.INV can be justified by the fact that agents' utility functions are von Neumann-Morgenstern types, which are unique up to positive affine transformations.

Nash [1950] showed that his solution is the unique solution satisfying PO, SY, S.INV and the following axiom.

Independence of Irrelevant Alternatives (IIA). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$, if $S^2 \subseteq S^1$, $d^2 = d^1$ and $F(S^1, d^1) \in S^2$, then $F(S^2, d^2) = F(S^1, d^1)$.

IIA requires that if an alternative has been judged superior to all others in some feasible set, then it should be judged superior to all others in any subset (to which it belongs) provided the disagreement point is kept constant.

Now we establish a characterization of the Nash solution by investigating the logical implication between IIA and our axioms.

Lemma 7. *Let F be a continuous linear solution. Then the solution satisfies IIA if and only if it satisfies INIR.*

Proof. It is clear that IR and IIA together imply INIR. To prove the converse statement, let $(S^1, d), (S^2, d) \in \Sigma^2$ be two problems such that $S^2 \subseteq S^1$ and $F(S^1, d) \in$

S^2. Now define the sequence of problems $\{(S^k, d^k)\}$ such that $S^2 \subseteq S^k \subseteq S^1$, $S^k \to S^2$, $d^k \in [d, F(S^1, d)[$ and $IR(S^k, d^k) = IR(S^1, d^k)$ for all k. By INIR, $F(S^k, d^k) = F(S^1, d^k)$ for all k. Since $d^k \in [d, F(S, d)[$ and F belongs to the linear family, $F(S^k, d^k) = F(S^k, d)$ and $F(S^1, d^k) = F(S^1, d)$ for all k. Altogether we have $F(S^k, d) = F(S^1, d)$ for all k. Since F is continuous, we conclude that $F(S^2, d) = F(S^1, d)$. Q.E.D.

Variants of the following theorem can be found in Chun and Thomson [1987b], Peters [1986b] and Peters and van Damme [1987]. Note that S-CONT and S.INV together imply d-CONT.

Theorem 8.[5] *A solution satisfies PO, INIR, S-CONT, D.Q-CAV and S.INV if and only if it is a weighted Nash solution.*

Remark 5. By dropping S-CONT from Theorem 8, the following solutions are permissible.

Definition. Given i, the i^{th} *benevolent dictatorial solution,* D^i, is defined by setting, for each $(S, d) \in \Sigma^2$, $D^i(S, d)$ equal to the point of $IR(S, d) \cap PO(S)$ preferred by agent i.

In fact, we can show that a solution satisfies PO, INIR, d-CONT, D.Q-CAV and S.INV if and only if it is a weighted Nash solution or a benevolent dictatorial solution. Since its proof is similar to that of Theorem 1 in Peters [1986b], we omit it.

Remark 6. D.Q-CAV in Theorem 8 can be replaced by the following axiom:

[5] This theorem can be generalized to n-person bargaining problems, as discussed in Chun and Thomson [1987b].

Restricted Disagreement Point Linearity (R.D.LIN). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all $\alpha \in [0,1]$, if $S^1 = S^2 \equiv S$, $\alpha F(S, d^1) + (1-\alpha)F(S, d^2) \in PO(S)$ and S is smooth at both $F(S, d^1)$ and $F(S, d^2)$, then

$$F(S, \alpha d^1 + (1-\alpha)d^2) = \alpha F(S, d^1) + (1-\alpha)F(S, d^2).$$

For details, we refer to Chun and Thomson [1987b].

It is well-known that the Nash solution is the only weighted Nash solution satisfying the symmetry axiom.

Corollary 2. *A solution satisfies PO, INIR, d-CONT, D.Q-CAV, S.INV and SY if and only if it is the Nash solution.*

5.4. Kalai-Rosenthal Solution

Finally, we discuss the Kalai-Rosenthal [1978] solution.

Definition. The *Kalai-Rosenthal solution, KR,* is defined by setting, for all $(S, d) \in \Sigma^2$, $KR(S, d)$ be the maximal point of S on the segment connecting d and $a(S)$, where, for each i, $a_i(S) \equiv max\{x_i | x \in S\}$.

To characterize the Kalai-Rosenthal solution, we introduce two additional axioms. For all $(S, d) \in \Sigma^2$, let $T(S_d) \equiv comp\{(d_1, a_2(S)), (a_1(S), d_2)\}$.

Independence of Strongly Individually Rational Outcome (ISIR). For all $(S, d) \in \Sigma^2$ and for all $x \in \Re^2$, if $S = comp\{IR(S, d)\}$, $x \leq d$ and $F(T(S_d), x) = F(T(S_d), d)$, then $F(S, x) = F(S, d)$.

Strict Disagreement Point Monotonicity (S.D.MON). For all $(S^1, d^1), (S^2, d^2) \in \Sigma^2$ and for all i, j such that $i \neq j$, if $S^1 = S^2$, $d_i^1 = d_i^2$, $d_j^1 < d_j^2$ and $a(S) \notin S$, then $F_j(S^2, d^2) > F_j(S^1, d^1)$.

ISIR, introduced by Peters [1986b], is interpreted as a weak form of path independence. S.D.MON, introduced by Livne [1987], requires that if an agent's utility at the disagreement point increases while the other's remains fixed, then the agent should gain strictly. This is a strengthening of the condition introduced by Thomson [1987].

A variant of the next theorem is in Peters [1986b].

Theorem 9. *A solution satisfies PO, IR, d-CONT, D.Q-CAV, SY, S.INV, ISIR and S.D.MON if and only if it is the Kalai-Rosenthal solution.*

Proof. It is clear that KR satisfies all eight axioms. Conversely, let F be a solution satisfying the eight axioms. From Theorem 2, it is a Pareto-optimal member of the linear solutions. Now by borrowing the proof of Theorem 4.1 in Peters [1986b], we can obtain the desired conclusion. Q.E.D.

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