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GENERALIZED INVERSE ESTIMATION OF PARTIAL ELASTICITIES OF SUBSTITUTION: A COMPLETE DEMAND SYSTEM WITH EXCLUSION RESTRICTIONS AND AUTOREGRESSIVE ERRORS

BY

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Abstract

In the literature on the estimation of singular systems of equations, researchers typically obtain maximum likelihood estimates after dropping an equation from the system. This approach appeals to Barten's (1969) invariance result. This paper contains the application of the generalized inverse estimation procedure of Dhrymes (Econometric Theory, forthcoming), where no equations of the system are dropped during estimation and where autoregressive errors are present. Non-adhoc exclusion restrictions are introduced through Leser's Transformation of the linear Expenditure system. Relying on quarterly data of the U.S., Allen partial elasticities are estimated. The estimated elasticities of substitution and expenditure elasticities are of reasonable magnitudes and are comparable to results in the literature. The estimated elasticities of substitution indicate well behaved indifference surfaces that satisfy theoretical curvature conditions.

This paper is a revision of an essay from the authors Ph.D. Dissertation.

1.1 Introduction

Dhrymes (Econometric Theory, forthcoming) developed an econometric procedure which estimates systems of singular equations, where autoregressive errors are present. The procedure critically relies on the generalized inverse of the contemporaneous covariance matrix. In the procedure, no equation of the system is deleted and all restrictions are imposed. This procedure is applied to Leser's transformation of the Linear Expenditure System ("LES"), where exclusion restrictions and autoregressive errors are present. Various lag structures are investigated. Using quarterly United States data for the period 1975 to 1985, positive partial elasticities of substitution are obtained without imposing curvature restrictions on indifference surfaces. The estimated elasticities of substitution indicate well behaved indifference surfaces.

This paper is organized into five sections. First, the literature on the consumer allocation problem is briefly surveyed in order to provide perspective. Second, the LES is reviewed and the problems encountered when the procedure is attempted are discussed. Third, Leser's transformation of the LES is presented, and a proposition permitting the application of the procedure is developed. Fourth, data and estimation results are examined. Finally, comparisons of the estimated results with selected literature are made.

1.2 Perspective

A widely used model for analyzing the consumer allocation problem is the Linear Expenditure System ("LES"). Stone (1954), Parks (1969), Pollak and Wales (1969, 1979), Goldberger and Gamaletsos (1970), Deaton (1975), Lluch and Powell (1975), Lluch, Powell, and Williams (1977), Lewis and Anderson (1989), and others have applied the LES to the consumer allocation problem. These studies are individual country studies and comparative studies of various countries.

Goodness of fit and flexibility considerations have lead to the investigation of flexible functional forms such as the translog, generalized Leontief, Almost Ideal Demand System, inverse translog - LES, and others. Lau, Lin, and Yotopolous (1978) and Pollak and Wales (1980) have estimated the translog functional form. Deaton and Muellbauer (1980) proposed and estimated the Almost Ideal Demand System. Blanciforti and Green (1983) estimated the Almost Ideal Demand System incorporating habit effects. Bush (1990) estimated the Almost Ideal Demand System with exclusion restrictions. Wang and Chern (1992) estimated the demand for rationed and nonrationed goods in an Almost Ideal Demand System. Diewert and Wales (1988) model the consumer allocation problem through the application of a normalized quadratic reciprocal indirect utility function and a normalized quadratic expenditure function.

In this literature, homogeneity and symmetry have been repeatedly tested. Deaton (1986) [1] indicates that the accumulated results from the literature reject symmetry and homogeneity restrictions. In addition, estimated demand systems, which are based on flexible functional forms, frequently fail to satisfy curvature conditions of microeconomic theory [2]. Indeed, the work of Diewert and Wales (1988) imposes curvature restrictions during estimation.

The "Australian" models of Leser (1961) and Powell (1969, 1974) approach the consumer allocation problem through a transformation of the Linear Expenditure System which permits the linear estimation of partial elasticites of substitution and marginal budget shares. In the "Australian" model homogeneity is imposed. The models incorporate a time trend, and restrictions are satisfied at means of observations. The results of the models appear reasonable [3]. However, a comparison between Powell's (1974) results and the LES results of Goldberger and Gamaletsos (1970) shows disparate results for marginal budget shares and elasticities of substitution.

Since the contemporaneous covariance matrix of contemporaneous errors is singular for demand systems addressing the consumer allocation problem, the estimation procedure generally followed is the application of the maximum likelihood technique after dropping an equation. This approach is based on Barten's (1969) invariance result and is applied, in the presence of autoregressive errors, as indicated by Berndt and Savin (1975). However, Powell (1974) estimates Leser's transformation of the LES by applying a feasible Aitken-Zellner estimator after dropping one equation even though exclusion restrictions are present. In addition, Wellman (1992) and other authors have applied Zellner's iterative seemingly unrelated regression technique after dropping an equation from the system.

The issues and difficluties associated with Barten and Powell's estimation procedures have been thoroughly analyzed by Dhrymes (1987a, 1987b). Using data and models of Berndt and Savin (1975), Bush (1990) conducted an empirical comparison between the conventional procedure of dropping an equation and the procedure which retains all equations and relies on the generalized inverse of the covariance matrix. The procedure utilizing the generalized inverse and retaining all equations generally increases computational accuracy which is reflected in smaller estimated standard errors for estimated parameters.

1.3 The Linear Expenditure System

The solution to the problem

$$\text{max.} \prod_{k=1}^{n}(x_k - \gamma_k)^{\beta_k} \text{ subject to } \Sigma_{k=1}^{n} p_k x_k = \mu \tag{1.1}$$

where x_k denotes the consumption of the kth good, p_k denotes the price of the kth good, and μ denotes total expenditure; yields a system of demand equations. The corresponding expenditure functions are

$$p_k x_k = p_k \gamma_k + \beta_k(\mu - \Sigma_{i=1}^{n} p_i \gamma_i), \quad k = 1, \ldots, n. \tag{1.2}$$

If the $\gamma_k > 0, k = 1, \ldots, n$ and $\mu - \Sigma_{i=1}^{n} p_i \gamma_i > 0$, the γ_k's are described as subsistence (permanent) levels of purchasing of various goods. The supernumerary income is divided among goods in proportions $(\beta_1, \ldots, \beta_n)$. However, Pollak (1971) indicates that the traditional interpretation of the $\gamma' s$ as subsistence parameters can be abandoned when all goods are consumed in positive quantities.

In the Linear Expenditure System specification, all variables are contained in all equations, and the LES is nonlinear in parameters. Neither the LES or various flexible functional forms are the ideal choice for applying the estimation technique of Dhrymes, since the models are nonlinear in parameters and exclusion restrictions are not conveniently imposed without resorting to adhoc theoretical specifications. Therefore, the "Australian" model of Powell, which is Leser's transformation of the Linear Expenditure System, is estimated with exclusion restrictions and autoregressive errors.

1.4 Leser's Transformation

The elasticity of substitution between commodity i and j is denoted σ_{ij}.

$$\sigma_{ij} = \frac{d(\frac{x_1}{x_2})}{(\frac{x_1}{x_2})} / \frac{d(MRS)}{MRS}, \tag{1.3}$$

where $MRS = -\frac{dx_1}{dx_2}$. Allen partial elaticities of substitution are generalizations of the two dimensional elasticity of substitution.

Leser (1961) expressed the LES as a linear function of the partial elasticities of substitution, σ_{ij}'s, and the β_k's which are the fixed proportions of supernumerary income. Following Powell's (1974) [4] presentation on Leser's transformation, the expenditure on the ith commodity is

$$v_{it}^{*} = p_{it}x_{it} = \Sigma_{j=1}^{n} a_{ij}p_{jt} + \beta_i m_t \tag{1.4}$$

where p_{it} is the price of commodity i at time t. At time t, the quantity of commodity i is x_{it}, and m_t is total expenditure at time t. The parameter $a_{ij} = -\beta_i\gamma_j$ for $i \neq j$; and $a_{ii} = \gamma_i(1-\beta_i)$ for $i = j$. The objective of Leser's transformation is to express a_{ij} as a function of σ_{ij}.

Variables evaluated at their sample means are denoted by a degree sign. T is the number of sample observations. The expenditure on commodity i at the mean is $v_i^{\circ} = \Sigma_{t=1}^{T} v_{it}^{*}/T$. The price of commodity i at the mean is $p_i^{\circ} = \Sigma_{t=1}^{T} p_{it}/T$. Total expenditure at the mean is $m^{\circ} = (\Sigma_{t=1}^{T} m_t)/T$. The quantity of commodity i at the mean is $x_i^{\circ} = v_i^{\circ}/p_i^{\circ}$. The average budget share of commodity i at the mean is $\omega_i^{\circ} = (p_i^{\circ}x_i^{\circ})/m^{\circ} = v_i^{\circ}/m^{\circ}$. Finally, $\sigma_{ij}^{\circ} = \frac{\varepsilon_{ij}^{\circ}}{\omega_j^{\circ}}$ where ε_{ij}° is the income-compensated cross-price elasticity of consumption of i with respect to j [5]. Now, $\varepsilon_{ij}^{\circ} = \frac{\partial x_i^{\circ}}{\partial p_j^{\circ}}\frac{p_j^{\circ}}{x_i^{\circ}} + \frac{x_j^{\circ}}{x_i^{\circ}}\frac{\partial x_i^{\circ}}{\partial m^{\circ}}p_j^{\circ}$, where $\frac{\partial x_i^{\circ}}{\partial m^{\circ}} = \frac{\partial x_i}{\partial m}|_{x_i=x_i^{\circ},m=m^{\circ}}$.

Since $v_i = p_i x_i(p_1, \ldots, p_n; m)$ and $v_i = \Sigma_{j=1}^{n} a_{ij}p_j + \beta_i m$,

$$\frac{\partial v_i}{\partial m} = p_i\frac{\partial x_i}{\partial m} = \beta_i \Rightarrow \frac{\partial x_i^{\circ}}{\partial m^{\circ}} = \frac{\beta_i}{p_i^{\circ}}. \tag{1.5}$$

$$\frac{\partial v_i}{\partial p_j} = p_i\frac{\partial x_i}{\partial p_j} = a_{ij} \Rightarrow p_i^{\circ}\frac{\partial x_i^{\circ}}{\partial p_j^{\circ}} = a_{ij},\ i \neq j. \Rightarrow \frac{\partial x_i^{\circ}}{\partial p_j^{\circ}} = \frac{a_{ij}}{p_i^{\circ}},\ i \neq j.$$

$$\frac{\partial v_i}{\partial p_i} = p_i\frac{\partial x_i}{\partial p_i} + x_i = a_{ii} \Rightarrow p_i^{\circ}\frac{\partial x_i^{\circ}}{\partial p_i^{\circ}} + x_i^{\circ} = a_{ii} \Rightarrow \frac{\partial x_i^{\circ}}{\partial p_i^{\circ}} = \frac{a_{ii}}{p_i^{\circ}} - \frac{x_i^{\circ}}{p_i^{\circ}}.$$

Substituting into ε_{ij}°, $\varepsilon_{ij}^{\circ} = p_j^{\circ}\frac{a_{ij}}{v_i^{\circ}} + \beta_i\frac{v_j^{\circ}}{v_i^{\circ}},\ i \neq j$.

Since $\frac{\varepsilon_{ij}^{\circ}}{\omega_j^{\circ}} = \sigma_{ij}^{\circ}$, $\sigma_{ij}^{\circ} = \frac{p_j^{\circ}a_{ij}}{v_i^{\circ}\omega_j^{\circ}} + \beta_i\frac{v_j^{\circ}}{v_i^{\circ}\omega_j^{\circ}}, i \neq j. \Rightarrow a_{ij} = \frac{v_i^{\circ}\omega_j^{\circ}}{p_j^{\circ}}\sigma_{ij}^{\circ} - \beta_i\frac{v_j^{\circ}}{p_j^{\circ}},\ i \neq j.$

When $i = j$, $\varepsilon^{\circ}_{ii} = \frac{\partial x^{\circ}_i}{\partial p^{\circ}_i}\frac{p^{\circ}_i}{x^{\circ}_i} + \frac{\partial x^{\circ}_i}{\partial m^{\circ}}p^{\circ}_i$. Since $\frac{\partial x^{\circ}_i}{\partial m^{\circ}} = \frac{\beta_i}{p^{\circ}_i}$ and substituting, $\varepsilon^{\circ}_{ii} = p^{\circ}_i\frac{a_{ii}}{v^{\circ}_i} - 1 + \beta_i$.

Now, $\sigma^{\circ}_{ii} = \frac{\varepsilon^{\circ}_{ii}}{\omega^{\circ}_i} \Rightarrow a_{ii} = \frac{v^{\circ}_i\omega^{\circ}_i}{p^{\circ}_i}\sigma^{\circ}_{ii} + (1-\beta_i)\frac{v^{\circ}_i}{p^{\circ}_i}$. Thus,

$$v^{*}_{it} = \Sigma^{n}_{j\neq i}(\frac{v^{\circ}_i\omega^{\circ}_j}{p^{\circ}_j}\sigma^{\circ}_{ij} - \beta_i\frac{v^{\circ}_j}{p^{\circ}_j})p_{jt} + (\frac{v^{\circ}_i\omega^{\circ}_i}{p^{\circ}_i}\sigma^{\circ}_{ii} + (1-\beta_i)\frac{v^{\circ}_i}{p^{\circ}_i})p_{it} + \beta_i m_t. \tag{1.6}$$

Using (1.6) and the condition of homogeneity, i.e., $\Sigma^{n}_{j=1}\sigma^{\circ}_{ij}\omega^{\circ}_j = 0.$, it can be shown that

$$y_{it} = v^{*}_{it} - x^{\circ}_i p_{it} = \Sigma^{n}_{j\neq i}\sigma^{\circ}_{ij}\zeta_{ijt} + \beta_i(m_t - \Sigma^{n}_{j=1}x^{\circ}_j p_{jt}), \tag{1.7}$$

where $\zeta_{ijt} = \omega^{\circ}_i v^{\circ}_j(\frac{p_{jt}}{p^{\circ}_j} - \frac{p_{it}}{p^{\circ}_i})$.

1.5 Estimation of Leser's Transformation

In the procedure of Dhrymes, there is a condition requiring each variable of the system to appear in at least two equations. Except for the $m_t - \Sigma^{n}_{j=1}x^{\circ}_j p_{jt}$, no equation of system (1.7) contains a set of variables common to any other equation. However, through the requirements that the partial elasticities of substitution be symmetrical i.e., $\sigma_{ij} = \sigma_{ji}$ and that the constraints imposed on the parameters of the system satisfy the adding-up condition, each variable of system (1.7) appears in at least two equations of the system. Lemma 1 and Proposition 1 provide specific conditions where Leser's Transformation of the LES can be estimated using the generalized inverse estimation procedure.

Lemma 1.1 *Let* $\zeta_{ijt} = \omega_i^\circ \upsilon_j^\circ (\frac{p_{jt}}{p_j^\circ} - \frac{p_{it}}{p_i^\circ})$, *where* $\omega_i^\circ = (p_i^\circ x_i^\circ)/m^\circ$ *and*

$\upsilon_j^\circ = p_j^\circ x_j^\circ$. *Then,* $\zeta_{ijt} = -\zeta_{jit}$ *,i.e.,* $\zeta_{ij} = -\zeta_{ji}$, *where* $\zeta_{ij} = \begin{bmatrix} \zeta_{ij1} \\ \vdots \\ \zeta_{ijT} \end{bmatrix}$.

Proof.

$$\zeta_{ijt} = \frac{p_i^\circ x_i^\circ}{m^\circ} p_j^\circ x_j^\circ (\frac{p_{jt}}{p_j^\circ} - \frac{p_{it}}{p_i^\circ}) = -\frac{p_j^\circ x_j^\circ}{m^\circ} p_i^\circ x_i^\circ (\frac{p_{it}}{p_i^\circ} - \frac{p_{jt}}{p_j^\circ}).$$

$\zeta_{ijt} = -\omega_j^\circ \upsilon_i^\circ (\frac{p_{it}}{p_i^\circ} - \frac{p_{jt}}{p_j^\circ}) = -\zeta_{jit}$.

$\zeta_{ijt} = -\zeta_{jit} \Rightarrow \zeta_{ij} = -\zeta_{ji}$. q.e.d.

Proposition 1.1 *Suppose that we have the Leser Transformation of the LES*

$$y_{it} = \Sigma_{j\neq i}^{n} \sigma_{ij}^{\circ} \zeta_{ijt} + \beta_i \tilde{V}_t \tag{1.8}$$

where

$$\tilde{V}_t = m_t - \Sigma_{j=1}^{n} x_j^{\circ} p_{jt}, \text{ and } y_{it} = v_{it}^{*} - x_i^{\circ} p_{it}.$$

If

i. ζ_{ij} *replaces* ζ_{ji} *in the jth equation and* α *is the parameter associated with* ζ_{ij} *in the jth equation;*

ii. the adding-up constraints are imposed on the parameters of the system; and

iii. $\sigma_{ij}^{\circ} = \sigma_{ji}^{\circ}$ *(the elasticities of substitution are symmetrical)*

then, system (1.8) is preserved.

Proof.

Replacing ζ_{ji} by ζ_{ij} in equation j and recognizing that ζ_{ij} now occurs only in the ith and jth equations,

$$\begin{aligned} &\vdots \\ y_i &= \cdots + \sigma_{ij}^{\circ}\zeta_{ij} + \cdots + \sigma_{in}^{\circ}\zeta_{in} + \beta_i \tilde{V} \\ y_j &= \cdots + \alpha\zeta_{ij} + \cdots + \sigma_{jn}^{\circ}\zeta_{jn} + \beta_j \tilde{V} \\ &\vdots \end{aligned}$$

The adding-up constraints $\Rightarrow$ $\alpha = -\sigma_{ij}^{\circ}$ and $\Sigma_{j=1}^{n}\beta_j = 1$. Thus,

$$\vdots$$

$$
\begin{aligned}
y_i &= \cdots + \sigma^\circ_{ij}\zeta_{ij} + \cdots + \sigma^\circ_{in}\zeta_{in} + \beta_i\tilde{V} \\
y_j &= \cdots - \sigma^\circ_{ij}\zeta_{ij} + \cdots + \sigma^\circ_{jn}\zeta_{jn} + \beta_j\tilde{V} \\
&\vdots
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
&\vdots \\
y_i &= \cdots + \sigma^\circ_{ij}\zeta_{ij} + \cdots + \sigma^\circ_{in}\zeta_{in} + \beta_i\tilde{V} \\
y_j &= \cdots + \sigma^\circ_{ij}\zeta_{ji} + \cdots + \sigma^\circ_{jn}\zeta_{jn} + \beta_j\tilde{V} \\
&\vdots
\end{aligned}
$$

by Lemma 1.

Since $\sigma_{ij} = \sigma_{ji}$

$$
\begin{aligned}
&\vdots \\
y_i &= \cdots + \sigma^\circ_{ij}\zeta_{ij} + \cdots + \sigma^\circ_{in}\zeta_{in} + \beta_i\tilde{V} \\
y_j &= \cdots + \sigma^\circ_{ji}\zeta_{ji} + \cdots + \sigma^\circ_{jn}\zeta_{jn} + \beta_j\tilde{V} \\
&\vdots
\end{aligned}
$$

which is system (1.8). q.e.d.

1.6 Model Specification

Following Pollak and Wales (1969) and Powell (1974), food, clothing and shoes, shelter, and other are the four commodity categories analyzed. Proposition 1 provides the following specification.

$$
\begin{aligned}
y_F &= \sigma^{\circ}_{FC}\zeta_{FC} + \sigma^{\circ}_{FS}\zeta_{FS} + \sigma^{\circ}_{FO}\zeta_{FO} + \eta_1 Trend + \beta_F\tilde{V} + u_{Food} \\
y_C &= \alpha_1\zeta_{FC} + \sigma^{\circ}_{CS}\zeta_{CS} + \sigma^{\circ}_{CO}\zeta_{CO} + \eta_2 Trend + \beta_C\tilde{V} + u_{Clothing} \\
y_S &= \alpha_2\zeta_{FS} + \alpha^{\circ}_3\zeta_{CS} + \sigma^{\circ}_{SO}\zeta_{SO} + \eta_3 Trend + \beta_S\tilde{V} + u_{Shelter} \\
y_O &= \alpha^{\circ}_4\zeta_{FO} + \alpha^{\circ}_5\zeta_{CO} + \alpha^{\circ}_6\zeta_{SO} + \eta_4 Trend + \beta_O\tilde{V} + u_{Other}
\end{aligned}
$$

$y_{it} = v^*_{it} - x^{\circ}_i p_{it}$,

$i = Food,\ Clothing,\ Shelter,\ Other$ and

FC: Food, Clothing FS: Food, Shelter FO: Food, Other

CS: Clothing, Shelter CO: Clothing, Other SO: Shelter, Other.

$$
y_i = \begin{bmatrix} y_{1i} \\ \vdots \\ y_{Ti} \end{bmatrix}, \text{ e.g., } y_F = \begin{bmatrix} y_{1F} \\ \vdots \\ y_{TF} \end{bmatrix}. \ \tilde{V}_t = m_t - \Sigma^n_{j=1} x^{\circ}_j p_{jt},\ \tilde{V} = \begin{bmatrix} \tilde{V}_1 \\ \vdots \\ \tilde{V}_T \end{bmatrix}.
$$

Food, clothing and shoes, shelter, and other are denoted commodities 1,2,3, and 4, respectively. $\beta_{.i}$ is a 5x1 vector containing the coefficients of equation i. The number of commodities is denoted by m, and $\beta = (\beta'_{.1}, \beta'_{.2}, \beta'_{.3}, \beta'_{.4})'$. The selection matrix for equation i is S_{i1}, and S_{i1} is a permutation of 5 of the columns of I_8 . The system can be written $y_{t\cdot} = X_{t\cdot}B + u_{t\cdot}$, where $y_{t\cdot} = [y_{t1}, y_{t2}, y_{t3}, y_{t4}], X_{t\cdot} = [\zeta_{12t}, \zeta_{13t}, \zeta_{14t}, \zeta_{23t}, \zeta_{24t}, \zeta_{34t}, Trend_t, \tilde{V}_t]$, and $S_{i1}\beta_{\cdot i} = b_{\cdot i}$. The diagonal matrix $S_1 = diag(S_{11}, S_{21}, S_{31}, S_{41})$.

Dhrymes (Econometric Theory, forthcoming) provides the the generalized inverse esimation procedure when autoregressive errors are present. The restrictions on B can be written as $R_1\beta = r_1$, where $R_1 = (e' \otimes I_G)S_1$. The matrix I_G is an identity matrix of order $G = 8$. In addition, e is a mx1 matrix containing ones, and $Vec(B) = S_1\beta$. The matrix r_1 is a Gx1 and has zeros in all positions except the Gth position which contains 1. With autoregressive errors, $u_{t.} = u_{t-1.}H + \xi_{t.}$, where $\{\xi'_{t.}, t = 1, 2, \ldots\}$ is a sequence of i.i.d random vectors. Now, $E\xi'_{t.} = 0$, and $Cov(\xi'_{t.}) = \Sigma$. If $m_i \leq m$ lags appear in the ith equation, S_{i2} is defined to be a permutation of m_i of the columns of I_m. Thus, $h_{.i} = S_{i2}\gamma_{.i}$, where $\gamma_{.i}$ contains the elements of $h_{.i}$ not known to be zero. Now, $S_2 = Diag(S_{12}, \ldots, S_{m2})$, and $vec(H) = S_2\gamma$, where $\gamma = (\gamma'_{.1}, \ldots, \gamma'_{.4})'$. The adding-up restrictions imply restrictions on H, and the restrictions on H are expressed as $R_2\gamma = r_2$, where $R_2 = (e' \otimes I^*_{m-1})S_2$ and $I^*_{m-1} = [I_{m-1}, -e_{m-1}]$. The identity matrix I_{m-1} is $(m-1)$x$(m-1)$, and e_{m-1} is an $(m-1)$x1 vector of ones. The matrix r_2 is an appropriately dimensioned matrix of zeros.

The estimators are derived from minimizing

$$L = (y - w)'(\Sigma_g \otimes I_T)(y - w) + 2\lambda'_1(R_1\beta - r_1) + 2\lambda'_2(R_2\gamma - r_2)$$

where $y = vec(Y)$ and where $w = [I_m \otimes (Y_{-1} - X_{-1}B)]S_2\gamma + (I_m \otimes X)S_1\beta$. The vectors of lagrangian multipliers are λ_1 and λ_2.

1.7 Data

The categories of analysis are food, clothing, which includes shoes, shelter, and other. Shelter excludes expenditures for household operations which are contained in the other category. The other category contains all other commodity groups. Quarterly data from 1975 to the first quarter of 1985 were available on total expenditures and implicit price deflators for food, clothing-shoes, shelter, and other commodites. The data are unpublished data on personal consumption expenditures from Appendix II of the National Income and Product Accounts of the United States [6].

All expenditures were in 1972 dollars and the implicit price deflators used 1972 as the base year. The implicit price deflators for the other category were constructed by creating a weighted sum of implicit price deflators of all commodities except food, clothing-shoes, and shelter. For a particular commodity in the other category, the weight was the ratio of the expenditures on the commodity to the total expenditure for the category.

Quarterly expenditures per capita were calculated by dividing quarterly consumption expenditures by U.S. resident population. The source of the quarterly population data was the Current Population Reports Series P-25, N 1006, Table 4. y_{it} and ζ_{ijt} were constructed using the implicit price deflators and the per capita expenditure data. A trend variable is included in the specification of the model. The trend variable is intended to capture changing consumer tastes [7].

The estimation was performed on an IBM Personal Computer using APL and software coded by the author.

1.8 Results

There are numerous lag structures that can be analyzed using the generalized inverse procedure of Dhrymes. Three lag structures are examined. The first lag structure eliminates all lags i.e., autoregressive errors do not exist. The second lag structure requires that each equation contain only its' own lag. The third lag structure examines the effects of various lag combinations on the elasticities of substitution. The results are presented in Tables 1,2, and 3.

Two criteria were used to control the iteration process. If the parameters converged, then the estimation process was discontinued; or if the absolute value of the difference between the current iteration's minimand and the previous iteration's minimand was less than or equal to 0.001, then the iteration process was discontinued. Model 1 converged after 25 iterations. The marginal budget shares and the elasticities of substitution were all positive. Minus twice the natural log of the likelihood ratio is 9972.7528 which indicates that the regression is significant.

Model 2 introduced autoregressive errors. The Durbin-Watson statistic for each equation indicates autocorrelation. A lag structure was introduced, where each equation contained its' own lagged error. The marginal budget shares and elasticities of substitution for all commodities are positive and statistically significant. Positive autocorrelation is present as indicated by a t-statistic of 12.78 for the autoregressive parameter. The chi-square statistic for testing the null hypothesis $\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = 0$ against the alternative hypothesis $\begin{pmatrix} \beta \\ \gamma \end{pmatrix} \neq 0$ is 8628.67 indicating the overall significance of the regression parameters. The chi- square statistic for testing the null hypothesis $\gamma = 0$ against the alternative hypothesis $\gamma \neq 0$ i.e., testing for autocorrelation, is 163.32246 implying the presence of autoregressive errors. These results suggest that Leser's transformation of the LES should be estimated with autoregressive errors.

In Model 3 an alternative lag structure is examined. All marginal budget shares and elasticities of substitution are positive and significant. Each autoregressive parameter is significant. The chi-square statistic for testing the null hypothesis $\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = 0$ against the alternative hypothesis $\begin{pmatrix} \beta \\ \gamma \end{pmatrix} \neq 0$ is 44494.441. In addition, the chi-square statistic for testing the null hypothesis $\gamma = 0$ against the alternative hypothesis $\gamma \neq 0$ is 273.36 which indicates the significance of the autoregressive parameters. Comparing the square of the simple correlation between predicted and actual values of Models 2 and 3, Model 3 suggests that the choice of lag structure effects the predictive ability of the model and the magnitude of the parameters.

Table 4 contains expenditure elasticities for the three models. The expenditure elasticities of shelter reflect the fact that household operations expenditures are excluded from the shelter category. The expenditure elasticities are sensitive to the lag structure selected, however the expenditure elasticities of Model 3 are within reasonable range of estimates found in the literature.

1.9 Selected Comparision

Powell (1974) reports partial elasticity of substitution estimates for the United States. In that work, the elasticity of substitution of Food-Other is negative, and the elasticity of substitution of Clothing-Shelter is negative. Powell's results and the literature indicate that the estimates of partial elasticities of substitution can be the wrong sign, since negative elasticities of substitution can indicate misbehaved indifference surfaces. A basic question is whether negative signs for the elasticities of substitution obtained by Powell (1974) is due to econometric technique or the underlying data. To address this question, the technique of Dhrymes is applied to data that is comparable to the data used by Powell. In addition, the econometric method of Powell is applied to the quarterly United States data.

Powell indicates that the source of his data is official statistics, however the exact data source is not identified. Therefore, official National Income and Product Account data for the period 1955 to 1967 were obtained. Constant and current expenditure data on food, clothing and shoes, housing, and all other commodities were from Appendix II of the National Income and Product Accounts of the United States, 1953-1984. Implicit price indices were constructed by dividing current expenditures by constant expenditures. The population series was obtained from Table No. 2 of the Statistical Abstract of the United States (1984), and per capita expenditures constructed. The implicit price deflators used 1972 as the base year, and all expenditures were in 1972 dollars. Following Powell (1974), per capita expenditures, and the implicit price indices were deflated by an index of overall price level to remove inflationary trends. The index employed was the Consumer Price Index taken from Table No. 809 of the Statistical Abstract of the United States (1984). From this data, independent and dependent variables from Leser's transformation were constructed. The technique of Dhrymes was applied to this comparable Powell data. The results appear in Table 5.

Except for the elasticity of substitution between shelter and "other", the elasticities of substitution estimated by Powell are less than twice their standard deviations. All elasticities of substitution estimated through the generalized inverse technique were less than twice their aysmptotic standard deviations. Only the elasticity of substitution between housing and "other" was negative.

Powells estimation procedure did not consider autoregressive errors, and the results in Table 5 of the new technique are not based on an autoregressive error process. However, the comparable data and model were estimated with autoregressive errors. The sign of the elasticity of substitution between housing and other remained negative under all estimated lag structures, while all other signs remained positive.

Powell (1974) employed a feasible Aitken estimator, after imposing symmetry restrictions. Powell [8] writes the system as:

$$\begin{bmatrix} y_F \\ y_C \\ y_S \end{bmatrix} = \begin{bmatrix} \tilde{V} & 0 & 0 & : & \zeta_{F,C} & \zeta_{F,S} & \zeta_{F,O} & 0 & 0 & 0 \\ 0 & \tilde{V} & 0 & : & \zeta_{C,F} & 0 & 0 & \zeta_{C,S} & \zeta_{C,O} & 0 \\ 0 & 0 & \tilde{V} & : & 0 & \zeta_{S,F} & 0 & \zeta_{S,C} & 0 & \zeta_{S,O} \end{bmatrix} \begin{bmatrix} \beta_F \\ \beta_C \\ \beta_S \\ \sigma^{\circ}_{F,C} \\ \sigma^{\circ}_{F,S} \\ \sigma^{\circ}_{F,O} \\ \sigma^{\circ}_{C,S} \\ \sigma^{\circ}_{C,O} \\ \sigma^{\circ}_{S,O} \end{bmatrix} + \begin{bmatrix} u_F \\ u_C \\ u_S \end{bmatrix}$$

This system will satisfy the adding-up constraints residually. However, the information contained in the adding-up constraint is not imposed during estimation of $\beta_F, \beta_C, and \beta_S$ i.e., $\beta_F + \beta_C + \beta_S + \beta_O$ is not constrained to be one during estimation of β_F, β_C, and β_S. Dhrymes and Schwarz (1987a) have shown that when different variables appear in different equations and the constraints on parameters required by the adding-up restrictions are not imposed, the estimation results depend on the equation deleted.

The difference between Powell's results and the estimation results obtained through the application of the procedure, which employs the generalized inverse, lies partially in the 1955-1967 data as indicated by the negative elasticity of substitution between housing and other. The other component which explains the differing results lies in the estimation technique employed by Powell. This is illustrated through the application of the technique of Powell to the quarterly United States data for 1975:1 to 1985:1. The results from the application of the technique of Powell are erroneous. For example, the marginal budget share of food is -0.027; clothing has a marginal budget share of 0.0357; housing has a marginal budget share of -0.0315; and other has as marginal budget share of 1.0228. The elasticity of substitution between clothing and shelter is negative. In addition, two of the elasticities of substitution are insignificant.

Given the differences between the econometric procedures and data underlying Table 5 and the procedure and quarterly data of this paper, the general magnitude (absolute value) of the estimates in Table 5 and the estimates obtained under various lag structures of Tables 1, 2, and 3 are not extraordinarily different.

The work of Wales (1977), Caves and Christensen (1980), Diewert and Wales (1988) and others has focused on flexible functional forms and curvature restrictions required by theory. Flexible functional forms often violate theoretical curvature conditions over price and income data. Diewert and Wales (1988) imposed curvature restrictions during estimation. However, the appearance of the procedure utilizing the generalized inverse requires a re-examination of the curvature issue. Since most applications of flexible functional forms continue to employ the Barten (1969) invariance result as applied by Berndt and Savin (1975) in the presence of autoregressive errors, the estimation issue must be separated from the curvature issue. Indeed, for all lag structures examined, estimated elasticities of substitution are positive for quarterly United States data. The positivity of the elasticities of substitution indicates well-behaved indifference surfaces. In addition, the square of the simple correlation between actual and predicted expenditures is high for all commodity categories.

1.10 Conclusion

The econometric procedure of Dhrymes for estimating systems of singular equations with autoregressive errors was employed to estimate Leser's Transformation of the Linear Expenditure System. Exclusion restrictions were imposed. For various lag structures significant and positive elasticities of substitution were obtained. Autoregressive errors are present and significant. For the decade 1975 to 1985, the indifference surfaces in the United States were well-behaved.

APPENDIX 1

Hypothesis Testing and The Generalized Inverse

Dhrymes [9] has shown that (A.1)

$\sqrt{T}\left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix}-\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right)$ is asymptotically distributed as $N(0, B_{11})$. B_{11} is singular due to the adding-up restrictions. To test H_0: $\begin{pmatrix}\beta\\ \gamma\end{pmatrix} = 0$ against H_1 $\begin{pmatrix}\beta\\ \gamma\end{pmatrix} \neq 0$ a chi-square statistic is constructed as follows:

$B_{11} = D_1\Lambda_1 D_1'$, where Λ_1 is a diagonal matrix containing the nonzero (decreasing in order of magnitude) characteristic roots of B_{11}. D_1 is a matrix containing the orthonormal characteristic vectors of the corresponding nonzero roots of B_{11}.

(A.2)

$$\sqrt{T}D_1'\left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix}-\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right) \sim N(0, D_1'D_1\Lambda_1 D_1'D_1) = N(0, \Lambda_1)$$

Let $\Lambda_1^{-1} = P'P$.

$$y = \sqrt{T}PD_1'\left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix}-\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right) \sim N(0, P\Lambda_1 P') = N(0, PP^{-1}P'^{-1}P') = N(0, I).$$

$$y'y \sim \mathcal{X}^2$$

with degrees of freedom equal to the number of parameters.

(A.3)

$$y'y = T\left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix} - \begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right)' D_1 P' P D_1' \left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix} - \begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right)$$

Since $D_1 P' P D_1' = (B_{11})_g$,

$$\mathcal{X}^2 = T\left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix} - \begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right)' (B_{11})_g \left(\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix} - \begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right).$$

To test $H_0 : \gamma = 0$ against $H_1 : \gamma \neq 0$, the lower $\Sigma m_i \text{x} \Sigma m_i$ submatrix of B_{11} is selected. Denote the submatrix $B_{11}^{(2)}$. A chi-square statistic is formed based on $(B_{11}^{(2)})_g$.

Table 1

MODEL 1

H = 0

	MARGINAL BUDGET SHARE	ELASTICITIES OF SUBSTITUTION CLOTHING	SHELTER	OTHER	TREND
FOOD	0.15289	3.34542	0.46233	1.84132	-2.88369
STD*	0.05067	3.47439	1.42192	1.12639	2.01209
CLOTHING	0.14335		1.53011	2.80762	-1.06921
STD	0.07806		2.38733	3.01763	2.16452
SHELTER	0.12918			1.7624	-0.57432
STD	0.03571			0.84106	1.38752
OTHER	0.57457				4.52721
STD	0.10769				3.32079

SQUARED SIMPLE CORRELATION COEFICIENTS**

FOOD	0.997552
CLOTHING	0.93079
SHELTER	0.998498
OTHER	0.999483

MINUS TWICE THE ln OF THE LIKELIHOOD RATIO 9972.7528
*STD: ESTIMATED STANDARD ERROR
** SQUARE OF THE CORRELATION BETWEEN THE PREDICTED AND THE ACTUAL VARIABLE

Table 2

MODEL 2

	MARGINAL BUDGET SHARE	ELASTICITIES OF SUBSTITUTION CLOTHING	SHELTER	OTHER	TREND
FOOD	0.14457	2.95635	0.99539	1.96703	-3.93562
STD	0.017389	0.79993	0.32112	0.23764	0.86234
T-ST	8.31	3.7	3.1	8.28	-4.56
CLOTHING	0.13283		3.60594	2.20272	-1.64941
STD	0.01457		0.84898	0.50651	0.70561
T-ST	9.12		4.25	4.35	-2.34
SHELTER	0.07779			0.77578	-1.902
STD	0.01188			0.23646	0.64233
T-ST	6.55			3.28	-2.96
OTHER	0.64481				7.48703
STD	0.02676				1.17897
T-ST	24.1				6.35

$h_{11} = \rho = 0.77815$
STD 0.06089
T-ST 12.78

	r-square*	DURBIN WATSON
FOOD	0.998352	1.28587
CLOTHING	0.915109	1.58914
SHELTER	0.999332	1.44727
OTHER	0.999634	1.56147

CHI SQUARE FOR TESTING HO: $\gamma = 0$ AGAINST H1: $\gamma \neq 0$ IS 163.32246

* SQUARE OF THE CORRELATION BETWEEN THE PREDICTED AND ACTUAL VALUE.

** STD INDICATES STANDARD ERRORS

*** T-ST INDICATES T-STATISTIC

Table 3 - MODEL 3

	MARGINAL BUDGET SHARE	ELASTICITIES OF SUBSTITUTION CLOTHING	SHELTER	OTHER	TREND
FOOD	0.12836	3.43977	0.95062	1.81213	-4.10108
STD	0.01301	0.69175	0.30518	0.20767	0.63029
T-ST	9.87	4.97	3.11	8.73	-6.51
CLOTHING	0.13134		2.06454	2.33464	-1.07539
STD	0.01301		0.52216	0.48418	0.41726
T-ST	10.1		3.95	4.82	-2.58
SHELTER	0.07308			0.53027	-3.47699
STD	0.01208			0.24275	0.73202
T-ST	6.05			2.18	-4.75
OTHER	0.66722				8.65346
STD	0.02477				1.1015
T-ST	26.93				7.86

AUTOREGRESSIVE PARAMETERS

PARAMETER	ESTIMATE	STD	T-ST
h_{11}	0.69868	0.0629	11.11
h_{21}	0.35168	0.15727	2.24
h_{22}	0.347	0.14825	2.34
h_{33}	0.69868	0.0629	11.11
h_{43}	-0.13783	0.04851	-2.84
h_{44}	0.83651	0.05117	16.35

r-square*

FOOD	CLOTHING	SHELTER	OTHER
0.998322	0.932653	0.999428	0.999661

CHI SQUARE FOR TESTING H0:$\gamma = 0$ AGAINST H1:$\gamma \neq 0$ IS 273.36033

* SQUARE OF THE CORRELATION BETWEEN THE PREDICTED AND ACTUAL VALUE

** STD INDICATES STANDARD ERRORS

*** T-ST INDICATES T-STATISTIC

TABLE 4: EXPENDITURE ELASTICITIES

	MODEL 1	MODEL 2	MODEL 3
FOOD	0.79874	0.75527	0.67059
CLOTHING	1.70992	0.9279	1.56666
SHELTER	0.76902	0.4631	0.43505
OTHER	1.03197	1.15812	1.19837

Table 5
United States Data 1955-1967

Marginal Budget Shares

	Powell Result	New Technique
Food	.1172	.1641
Clothing	.1230	.1070
Housing	.0685	.0477
Other	.6913	.6813

Elasticities of Substitution

	Clothing	Housing	Other
Food			
Powell	1.120	1.853	-.394
New Technique	2.1644	.16747	1.5556
Clothing			
Powell		-3.455	.009
New Technique		1.2012	1.2891
Housing			
Powell			2.151
New Technique			-0.47971

FOOTNOTES

1. Angus Deaton, "Demand Analysis," in *Handbook of Econometrics* Volume 3, ed. Zvi Griliches and Micheal D. Intriligator (New York: North-Holland, 1986), 1791.

2. Terence J. Wales, "On the Flexibility of Flexible Functional Forms," *Journal of Econometrics* 5 (March 1977): 183- 193.

3. Alan Brown and Angus Deaton, "Surveys in Applied Economics: Models of Consumer Behavior," *The Economic Journal* 82 (December 1972): 1208.

4. Alan P. Powell, *Empirical Analytics of Demand Systems* (Massachusetts: Lexington Books, 1974), 62-66.

5. Ibid., 63.

6. The author is grateful to Professor Dhrymes for support and data.

7. The time trend begins in 1975:1, and the middle of the trend is 1980:1, where the trend is 0. The trend ends on 1985:1.

8. Powell, 68.

9. P. J. Dhrymes, Autoregressive Errors in Singular Systems of Equations, 1984; revised 1988, pp. 23-24, Discussion Paper No. 257, Department of Economics, Columbia University, New York; *Econometric Theory*, forthcoming.

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