General Conditions for Existence of Maximal Elements via the Uncovered Set

John Duggan

Working Paper No. 563
July 2011

General Conditions for Existence of Maximal Elements via the Uncovered Set

John Duggan*

July 8, 2011

Abstract

This paper disentangles the topological assumptions of classical results (e.g., Walker (1977)) on existence of maximal elements from rationality conditions. It is known from the social choice literature that under the standard topological conditions—with no other restrictions on preferences—there is an element such that the upper section of strict preference at that element is minimal in terms of set inclusion, i.e., the uncovered set is non-empty. Adding a condition that weakens known acyclicity and convexity assumptions, each such uncovered alternative is indeed maximal. A corollary is a result that weakens the semi-convexity condition of Yannelis and Prabhakar (1983).

1 Introduction

In the literature on binary relations and maximality, Walker (1977) provides a necessary condition for existence of maximal elements that is also sufficient under weak topological conditions. The usefulness of this result is that when the relation in question is viewed as a preference relation, it permits a non-vacuous theory of choice. The key condition, which I call the "finite-dominance property," is that there does not exist a finite set such that every

*Department of Political Science and Department of Economics, University of Rochester

alternative is strictly worse than some element of that set; equivalently, for every finite set, there is an alternative that is weakly preferred to all elements of that set. Although the latter condition is not totally transparent, it is separately implied by the intuitive and familiar conditions of acyclicity (e.g., Brown (1973), Bergstrom (1975)) and semi-convexity of preferences (Yannelis and Prabhakar (1983)). At the same time, a distinct literature has addressed the possibility of non-existence of maximal elements by recourse to a covering relation (or "majorization") induced by the initial relation. It is known from the social choice literature that under the standard topological conditions of Walker and others, a maximal element of the covering relation exists: there is an element such that the upper section of strict preference at that element is minimal in terms of set inclusion. Adding a condition that weakens Walker's finite-dominance property, I show that each such uncovered element is indeed maximal. In sum, this note employs the theory of the uncovered set to disentangle the standard topological assumptions from rationality assumptions on preferences and to generalize previous theorems on existence of maximal elements.

A variety of definitions of covering have been used (see Duggan (2011) for a review), and the one employed here is defined simply in terms of upper sections of the strict preference relation: for our purposes, an element is uncovered if and only if the set of strictly preferred elements is minimal according to set inclusion compared to all other elements. The identification of alternatives with minimal strict upper sections as plausible choices can be traced in spirit to Gillies (1959) and, later, to Fishburn (1977). Of course, any element at which the upper section of strict preference is empty—i.e., any maximal element of the strict preference relation—is uncovered in this sense, so the uncovered set is a natural extension of the idea of maximality. In contrast to maximal elements of strict preference, it is known that the uncovered set is non-empty under very general topological conditions: it is sufficient that at least one upper section of weak preference is compact and all lower sections of strict preference are open. Moreover, the uncovered elements are characterized by a "two-step" principle: z is uncovered if and only if for every x, either the upper sections of strict preference are equal at x and z, or there exists y such that z is weakly preferred to y, which is strictly preferred to x. This solution has received attention in social choice, where majority voting often fails to produce a Condorcet winner, and where the uncovered set has been proposed as an upper bound on political processes.

A trivial sufficient condition, under the above-mentioned topological conditions, for existence of maximal elements of the initial preference relation is then that every uncovered element is maximal with respect to strict preference. Perhaps surprisingly, this observation can be used to generalize Walker's finite-dominance property, while weakening the assumption that the set of alternatives is compact. The generalized rationality condition, which I call the "finite-subordination property," is that there does not exist a finite set containing a privileged element z, which is weakly preferred to all but one element of the set, and such that for every x, there is an element y of the finite set strictly preferred to x. Thus, the condition precludes the possibility of certain dominant finite sets, as in Walker (1977), but only those satisfying further internal structure; namely, one element of the set is weakly preferred to all but one other member of the set. The proof follows directly from a version the usual two-step principle (the latter due to Miller (1980)) based on weak topological background assumptions.

An easy corollary of the main result is an extension to relations that, instead, satisfy a weak version of semi-convexity. Imbedding the set of alternatives in a vector space and assuming it is convex, the latter condition dictates that no alternative belongs to the convex hull of the set of alternatives strictly preferred to it. I weaken this condition to subordinate convexity, which requires that no alternative can be written as a convex combination of a finite set of alternatives with the additional property that some alternative is weakly preferred to all or all but one element of that set. Thus, we extend a result of Yannelis and Prabhakar (1983) using the weaker subordinate convexity condition.

Following the early literature on maximality, a number of papers have maintained or strengthened the acyclicity condition of Brown (1973) and Bergstrom (1975), e.g., Mehta (1989), Campbell and Walker (1990), Peris and Subiza (1994), Tian and Zhou (1995), and Alcantud (2002). Nehring (1996) and Alcantud (2006) move to the non-binary framework and examine conditions on general choice functions. Horvath and Ciscar (1996) consider a partially ordered set of alternatives. In contrast, I maintain the binary approach without imposing any ordering structure, and I maintain (or slightly weaken) the usual topological conditions while exploring the extent to which the finite-dominance property and semi-convexity can be weakened.

2 Main Result

Given a set X, a relation B on X is **irreflexive** if for all $x \in X$, not xBx; it is **reflexive** if for all $x \in X$, xBx; it is **asymmetric** if for all $x, y \in X$, not both xBy and yBx; and it is **complete** if for all $x, y \in X$, either xBy or yBx (or both). Trivially, asymmetry implies irreflexivity, and completeness implies reflexivity. Given an element x, let $B(x) = \{y \mid (y, x) \in B\}$ be the **upper section** of B at x, and let $B^{-1}(x) = \{y \mid (x, y) \in B\}$ be the **lower section** of B at x. Define the **inverse** of B, denoted B^{-1}, by $B^{-1} = \{(y, x) \mid (x, y) \in B\}$; define the **dual** of B, denoted B^*, as the complement of the inverse, i.e., $B^* = \overline{(B^{-1})} = (\overline{B})^{-1}$; and define the **covering relation** of B, denoted $B^\bullet$, as $B^\bullet = \{(x, y) \mid B(x) \subseteq B(y)\}$. Note that $B = B^{**}$, that B is irreflexive if and only if B^* is reflexive, and that B is asymmetric if and only if B^* is complete. Moreover, the covering relation of B is transitive and reflexive, even if B fails to possess these properties. An element z is B-**maximal** if for all $x \in X$, xBz implies zBx. In case B is asymmetric, z is B-maximal if and only if there is no $x \in X$ such that xBz; and in case B is complete, z is B-maximal if and only if for all $x \in X$, zBx. An element z is $B^\bullet$-maximal if and only if the upper section $B(z)$ is minimal among $\{B(x) \mid x \in X\}$ with respect to set inclusion, in which case z is B-**uncovered,** and the set of such elements is the **uncovered set** of B.

Now let X be a topological space, let P be an irreflexive binary relation on X, and let $R = P^*$ denote the dual of P. When X is a set of alternatives available to a decision maker, we interpret P as a strict preference relation and R as a weak preference relation. We make use of the following assumptions: **(A1)** for all $x \in X$, the strict lower section $P^{-1}(x)$ is open, **(A2)** for some $x \in X$, the weak upper section $R(x)$ is compact, and **(A3)** for all $x \in X$, $R(x)$ is compact. The first lemma shows that the first two of these topological conditions—in the absence of transitivity properties of any sort—are sufficient for the existence of an uncovered element for the relation B. The result has been proved in slightly less generality by Banks, Duggan, and Le Breton (2002) and Duggan (2011).[1]

Lemma 1 *Assume (A1) and (A2). There exists a P-uncovered element.*

[1]Banks, Duggan, and Le Breton (2002) establish the result in the proof of their Theorem 3 assuming compactness of X, and Duggan (2011) states the result in Proposition 23 assuming asymmetry of P.

The proof follows immediately from results of Banks, Duggan, and Le Breton (2006). By (A1) and Proposition A4 of the latter paper, the covering relation $P^\bullet$ has closed upper sections (it is "upper semicontinuous" in the terminology of that paper). Moreover, the covering relation is transitive. By (A2), there is some x such that $R(x)$ is compact. Note that $P^\bullet(x) \subseteq R(x)$, for otherwise there exists $y \in P^\bullet(x) \setminus R(x)$, which implies $x \in P(y) \subseteq P(x)$, contradicting irreflexivity of P. Therefore, $P^\bullet(x)$ is compact for some x, and the existence of a $P^\bullet$-maximal element follows from Proposition A1 of the above-mentioned paper. This completes the proof of the lemma.

A straightforward observation (along the lines of Miller (1980)) is that an element z belongs to the uncovered set of P if and only if for all $x \in X$, either $P(x) = P(z)$ or there exists $y \in X$ such that $zRyPx$. Indeed, z is P-uncovered if and only if $P(z)$ is minimal among $\{P(x) \mid x \in X\}$, which holds if and only if for all $x \in X$, not $P(x) \subsetneq P(z)$, which holds if and only if for all $x \in X$, either $P(x) = P(z)$ or there exists $y \in P(x) \setminus P(z) = P(x) \cap R^{-1}(z)$, as claimed. I refer to this property as a "two-step principle." The next lemma employs topological conditions to establish one direction of a simplified two-step principle. Define a set $Y \subseteq X$ to be **finitely dominant** if Y is finite and for all $x \in X$, there exists $y \in Y$ with yPx. Given an element $z \in X$, we say P is **finitely subordinated** to z if there is a finitely dominant set Y with $z \in Y$ and such that there exists $v \in Y$ with $Y \setminus \{v\} \subseteq R^{-1}(z)$.[2] In words, P is subordinated to an element if that element belongs to a finitely dominant set and is weakly preferred to all but one member of that set.

Lemma 2 *Assume (A1) and (A2), and consider any $z \in X$ such that $R(z)$ is compact. If z is P-uncovered, then either z is P-maximal, or P is finitely subordinated to z.*

To prove the lemma, suppose z is P-uncovered and is not P-maximal, so there exists $v \in P(z)$. Consider any $x \in R(z) \setminus P^{-1}(v)$. Then $v \notin P(x)$, so $P(x) \neq P(z)$, and since z is P-uncovered, the two-step principle yields $y \in X$ with $zRyPx$. In particular, this implies

$$R(z) \setminus P^{-1}(v) \quad \subseteq \quad \bigcup_{y \in R^{-1}(z)} P^{-1}(y).$$

[2]Note the implication that vPz.

Since $R(z)$ is compact by assumption, and $P^{-1}(v)$ is open by (A1), it follows that $R(z) \setminus P^{-1}(v)$ is compact in the relative topology induced by the topology on X. Since $\{P^{-1}(y) \cap (R(z) \setminus P^{-1}(v)) \mid y \in R^{-1}(z)\}$ is an open covering of $R(z) \setminus P^{-1}(v)$ in the relative topology, by (A1), we conclude that there is a finite subset $\{y_1, \ldots, y_m\}$ of $R^{-1}(z)$ such that $R(z) \setminus P^{-1}(v) \subseteq \bigcup_{j=1}^{m} P^{-1}(y_j)$. Then P is finitely subordinated to z by way of the set $Y = \{z, v, y_1, \ldots, y_m\}$. This completes the proof of the lemma.

We can now proceed to the main theorem, which generalizes the result of Walker (1977) that, under (A1) and compactness of X, the absence of a finitely dominant set implies the existence of a P-maximal element: rather than precluding all finitely dominant sets, it is adequate to ban finitely dominant sets for which one member is weakly preferred to all but one other member, i.e., P is not subordinated to any element; and compactness of X can be weakened to compact upper sections of weak preferences.

Theorem 1 *The following properties are listed in decreasing strength.*

(C0) There is a P-maximal element.

(C1) There is no finite, externally stable cycle, i.e., there do not exist elements $y_1, \ldots, y_m \in X$ satisfying $y_1 P y_m$ and $y_{j+1} P y_j$ for all $j < m$, and such that for all $x \in X \setminus Y$, there exists $j \in \{1, \ldots, m\}$ with $y_j P x$.

(C2) There is no finitely dominant set.

(C3) There is no element $z \in X$ such that P is finitely subordinated to z.

Moreover, under (A1) and (A3), all of the above conditions are equivalent.

That (C0) implies (C1) is clear. That (C1) implies (C2) follows from the fact that the cycle $\{y_1, \ldots, y_m\}$ is finitely dominant. Similarly, if P is finitely subordinated to some element, then there is a finitely dominant set, so (C2) implies (C3). Now assuming (A1) and (A3), it suffices to prove that (C3) implies (C0). By Lemma 1, there is a P-uncovered element z, and by (A3), $R(z)$ is compact. Then Lemma 2 implies that either z is P-maximal or it is finitely subordinated to z. Since (C3) precludes the latter possibility, we conclude that z is P-maximal. This completes the proof of the theorem.

The absence of a finitely dominant set, which I term the **finite-dominance property,** has a well-known dual formulation:

(C2′) For every finite set $Y \subseteq X$, there exists $x \in X$ such that for all $y \in Y$, xRy.

Similarly, I term (C3) the **finite-subordination property,** which has the following dual formulation:

(C3′) For every finite set $Y \subseteq X$, if there exist $z, v \in Y$ with $Y \backslash \{v\} \subseteq R^{-1}(z)$, then there exists $x \in X$ such that for all $y \in Y$, xRy.

Equivalently, we may formulate this as follows:

(C3″) For all $z \in X$, all $v \in P(z)$, and all finite $W \subseteq R^{-1}(z)$, there exists $x \in X$ such that for all $y \in W \cup \{v, z\}$, xRy.

3 Implications

Assuming compactness of X, (A1) automatically implies (A3), and it implies that the set of P-maximal elements is closed. This suggests the following corollary.

Corollary 1 *Assume X is compact and (A1). If P possesses the finite-subordination property, then the set of P-maximal elements is nonempty and compact.*

Obviously, acyclicity of P implies (C1), and thus the finite-dominance property, and it is therefore sufficient for existence of a maximal element under (A1) and (A3). A different approach to verifying the finite-dominance property is to impose the structure of convexity. Assuming X is a convex subset of a vector space, we say P is **convex** if for all $x \in X$, $P(x)$ is convex; and it is **semi-convex** if for all $x \in X$, $x \notin \text{co}P(x)$. By irreflexivity of P, the former condition implies the latter, and the two are equivalent when P is a weak order (i.e., R is complete and transitive). That semi-convexity, assuming X is a Hausdorff topological vector space and assuming (A1), implies the finite-dominance property follows by an application of the KKM theorem of Knaster, Kuratowski, and Mazurkiewicz (1929). Adding compactness of X, existence of a P-maximal element follows, as established by Yannelis and Prabhakar (1983).

I use Theorem 1 to weaken the semi-convexity condition of Yannelis and Prabhakar (1983). To motivate the construction, I recast semi-convexity as follows: for all $x \in X$ and all finite $Y \subseteq P(x)$, it is not the case that $x \in \text{co}Y$. I now say P is **subordinate convex** if for all $x \in X$ and all finite $Y \subseteq P(x)$ such that there exists $z \in X$ with $|R^{-1}(z) \cap Y| \geq |Y| - 1$, it is not the case that $x \in \text{co}Y$. Subordinate convexity may allow an element to belong to the convex hull of its strict upper section in some situations, but it cannot be written as the convex combination of a finite set of strictly preferred elements if there is an element (possibly in the finite set but not necessarily) weakly preferred to all or all but one member of the set. The following theorem generalizes Theorem 4 of Sonnenschein (1971), which assumes compactness of X and convex upper sections of strict preference, and Theorem 5.1 of Yannelis and Prabhakar (1983), which replaces convexity with semi-convexity.[3] It is not logically related to Theorem 5.3 of the latter paper, which employs a different weakening of compactness of X than used here and which assumes strict upper sections are located within a given compact set.

Corollary 2 *Assume X is a convex subset of a Hausdorff topological vector space, and assume (A1). If P is subordinate convex, then it possesses the finite-subordination property. Adding (A3), there exists a P-maximal element.*

To prove the corollary, it suffices to deduce (C3′) from subordinate convexity of P under (A1). Consider any finite set $Y \subseteq X$ and elements $z, v \in Y$ with $Y \setminus \{v\} \subseteq R^{-1}(z)$. Index Y as $\{y_1, \ldots, y_m\}$, assuming $|Y| = m$ without loss of generality, and let V be the finite-dimensional vector space spanned by

[3] Furthermore, Sonnenschein (1971) considers choice from a finite-dimensional budget set, whereas Yannelis and Prabhakar (1983) consider a general subset of a Haudorff topological vector space. An implication of Schafer and Sonnenschein (1975) is a related result for the case in which the set of alternatives is finite-dimensional, and the strict preference relation has open graph and satisfies semi-convexity.

Y. As a finite-dimensional subspace of a Hausdorff tvs, V is homeomorphic to finite-dimensional Euclidean space (see Theorem 5.21 of Aliprantis and Border (2006)). By (A1), $\{R(y_1) \cap V, \ldots, R(y_m) \cap V\}$ is a family of closed subsets of V. Let $I \subseteq \{1, \ldots, m\}$ be any subset of indices, and consider any convex combination $x \in \mathrm{co}\{y_j \mid j \in I\}$. Since X is convex, we have $x \in X$. I claim that $x \in R(y_j)$ for some $j \in I$, for suppose otherwise. Then we have $x \in \bigcap_{j \in I} P^{-1}(y_j)$. But note that $|R^{-1}(z) \cap \{y_j \mid j \in I\}| \geq |I| - 1$, so subordinate convexity implies that $x \notin \mathrm{co}\{y_j \mid j \in I\}$, a contradiction. Therefore, the KKM theorem (see Lemma 17.43 of Aliprantis and Border (2006)) yields $x \in \bigcap_{j=1}^{m} R(y_j)$, fulfilling (C3′). This completes the proof of the corollary.

References

[1] J. Alcantud (2002) "Characterization of the Existence of Maximal Elements of Acyclic Relations," *Economic Theory,* 19: 407–416.

[2] J. Alcantud (2006) "Maximality with or without Binariness: Transfer-type Characterizations," *Mathematical Social Sciences,* 51: 182–191.

[3] C. Aliprantis and K. Border (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide,* Springer, Berlin.

[4] T. Bergstrom (1975) "Maximal Elements of Acyclic Relations on Compact Sets," *Journal of Economic Theory,* 10: 403–404.

[5] J. Banks, J. Duggan, and M. Le Breton (2002) "Bounds for Mixed Strategy Equilibria in the Spatial Model of Elections," *Journal of Economic Theory,* 103: 88–105.

[6] J. Banks, J. Duggan, and M. Le Breton (2006) "Social Choice and Electoral Competition in the General Spatial Model," *Journal of Economic Theory,* 31: 122–132.

[7] D. Brown (1973) "Acyclic Choice," Cowles Foundation Discussion Paper No. 360, Yale University.

[8] D. Campbell and M. Walker (1990) "Maximal Elements of Weakly Continuous Relations," *Journal of Economic Theory,* 50: 459–464.

[9] J. Duggan (2011) “Uncovered Sets,” unpublished paper.

[10] P. Fishburn (1977) “Condorcet Social Choice Functions,” *SIAM Journal of Applied Mathematics,* 33: 469–489.

[11] D. Gillies (1959) “Solutions to General Non-zero-sum Games,” in *Contributions to the Theory of Games IV,* A. Tuker and R. Luce, eds, *Annals of Mathematics Studies,* no. 40, Princeton University Press, Princeton.

[12] C. Horvath and J. Ciscar (1996) “Maximal Elements and Fixed Points for Binary Relations on Topological Ordered Spaces,” *Journal of Mathematical Economics,* 25: 291–306.

[13] B. Knaster, K. Kuratowski, and S. Mazurkiewicz (1929) “Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe,” *Fundamenta Mathematicae,* 14: 132–137.

[14] G. Mehta (1989) “Maximal Elements in Banach Spaces,” *Indian Journal of Pure and Applied Mathematics,* 20: 690–697.

[15] N. Miller (1980) “A New Solution Set for Tournaments and Majority Voting: Further Graph-theoretical Approaches to the Theory of Voting,” *American Journal of Political Science,* 24: 68–96.

[16] K. Nehring (1996) “Maximal Elements of Non-binary Choice Functions on Compact Sets,” *Economics Letters,* 50: 337–340.

[17] W. Shafer and H. Sonnenschein (1975) “Equilibrium in Abstract Economies without Ordered Preferences,” *Journal of Mathematical Economics,* 2: 345–348.

[18] H. Sonnenschein (1971) “Demand Theory without Transitive Preferences, with Applications to the Theory of Competitive Equilibrium,” in *Preferences, Utility, and Demand,* J. Chipman et al, eds, Harcourt Brace Jovanovich, New York.

[19] M. Walker (1977) “On the Existence of Maximal Elements,” *Journal of Economic Theory,* 16: 470–474.

[20] N. Yannelis and N. Prabhakar (1983)“Existence of Maximal Elements and Equilibria in Linear Topological Spaces,” *Journal of Mathematical Economics,* 12: 233–245.