Rochester Center for Economic Research

New variable-population paradoxes for resource allocation

William Thomson

Working Paper No. 575
September 2012

University of Rochester

New variable-population paradoxes for resource allocation

William Thomson*

May 2006; this version, September 21, 2012

Abstract

We identify previously unnoticed ways in which agents can strategically distort allocation rules, by affecting the set of "active" agents. (i) An agent withdraws with his endowment. (ii) He gives control of his endowment to someone else and withdraws. (iii) He invites someone in and let him use some of his endowment. (iv) He pre-delivers to some other agent the net trade that the rule would assign to that second agent if that second agent had participated. In (i) and (ii), he and his co-conspirator may end up controlling resources that allow them to reach higher welfare levels than they otherwise would. In (iii) and (iv), he may end up with a bundle that he prefers to the one he would have been assigned had he not engaged in the manipulation. We show that (i) the Walrasian rule is not "withdrawing-proof", nor "endowments-merging–proof, nor "endowments-splitting–proof", but that it is "pre-delivery–proof", and that (ii) canonical selections from the egalitarian-equivalence-in-trades solutions satisfy none of the properties.

Key-words: resource allocation rules; withdrawal-proofness; endowments-merging–proofness; endowments-splitting–proofness; pre-delivery–proofness.

*I thank Murat Atlamaz, Diego Dominguez, Biung-Ghi Ju, and Tarık Kara for helpful discussions, and two referees and the Associate Editor for their suggestions. An early version of this paper was presented at the First Annual Workshop on Social Choice, Maastricht, April 2006, and at SING3, Madrid, July 2007, and this version at Hitotsubashi University, October 2010 and Rice University, May 2011. Filename: newparadoxes

1 Introduction

We identify previously unnoticed ways in which a group of agents can distort allocation rules to their advantage. In each case, the manipulation has the effect of changing the set of "active" agents. We state axioms expressing the robustness of rules to each of these manipulations. We examine two central rules, and ask whether they satisfy these robustness requirements.

We consider two classes of economies that differ in the manner in which ownership rights over resources are specified. In a standard "economy with private endowments", each agent is endowed with a bundle of goods and allocation rules are designed to redistribute these bundles. In a "fair division problem", there is a social endowment of resources over which agents are understood to have equal rights, and rules are designed to distribute these resources in a way that "best" reflects these rights.

The first possibility of manipulation that we study in economies with private endowments is as follows: instead of participating, an agent withdraws with his endowment; the rule is applied without him; he then gets together with one of the agents who did participate. The two of them may end up controlling resources (the sum of the assignment to the agent who stayed and of the endowment of the agent who withdrew) that, when appropriately divided between them, make each of them at least as well off as he would have been without the manipulation, and at least one of them better off.

Two scenarios are possible in the context of the problem of fair division. First, when an agent withdraws, he may relinquish his rights on the social endowment altogether; as before, the rule is applied without him; some agent who stays may be assigned a bundle that can be redivided between the two of them so as to make each of them at least as well off as he would have been without the manipulation, and at least one of them better off.

Alternatively, the agent who withdraws leaves with an equal share of the social endowment; the rule is applied without him; the sum of the assignment to some agent who stays and the bundle taken with him by the agent who withdrew, an equal share of the social endowment, may be divided between the two of them so as to make each of them at least as well off as he would have been without the manipulation, and at least one of them better off.

The other possibilities of manipulation that we examine reflect other terms under which an agent may withdraw. Returning to economies in which agents are individually endowed, two agents can merge their endowments, and one of them withdraw; the rule is applied without this second agent; the agent who stays may be assigned a bundle that can be divided between the two of them in such a way that each of them is at least as well off as he would have been without the manipulation, and at least one of them is better off.

Symmetrically, an agent may split his endowment with some outsider—some agent with no endowment; the rule is applied and the guest then transfers his assignment to the agent who invited him in; the first agent may prefer his final assignment to what he would have received without the manipulation.

Finally, an agent may make a pre-delivery to some other agent of the trade the latter would be assigned if he participated; the rule is applied without the second agent; at his final assignment, the first agent may be better off that he would have been without the manipulation.

We examine the robustness to these various types of manipulation of two canonical selections from the correspondences that arguably are the most important in the literature on resource allocation in private good economies. The first such correspondence selects, for each economy, the allocations such that each agent finds his assignment at least as desirable as his endowment, that is, allocations that meet the "endowments lower bound". The other three are inspired by the literature on fairness. An allocation meets the "equal division lower bound" if each agent finds his assignment at least as desirable as an equal share of the social endowment. It is envy-free-in-trades (Foley, 1967; Schmeidler and Vind, 1972) if no agent prefers the net trade (the difference between final consumption and endowment) assigned to any other agent to the net trade assigned to him. Finally, it is "egalitarian-equivalence in trades" (a definition inspired by Pazner and Schmeidler, 1978) if there is a "reference trade vector" such that each agent finds his assigned trade indifferent to this reference vector.

The first of the two rules on which we focus is the Walrasian rule, or, when the issue is fair division, the equal-division Walrasian rule (defined by first specifying as a private endowment for each agent an equal share of the social endowment, and then applying the standard Walrasian definition). The Walrasian rule meets the endowments lower bound and it delivers envy-free trades. When operated from equal division, it meets the equal-division lower bound and it delivers envy-free allocations. Except when pre-delivery is concerned, the Walrasian rule is not robust. We establish these facts on domains of economies with homothetic preferences, and most of them also hold on domain of economies with quasi-linear preferences. The positive answer, concerning immunity to pre-delivery, holds under general assumptions on preferences of continuity, monotonicity, and convexity.

The most prominent selections from the egalitarian-equivalent-in-trades correspondence are defined by requiring that the reference trade be proportional to a particular vector, fixed once and for all, independently of preferences. Their prominence stems from the fact that they satisfy a wide array of relational fairness properties. It is therefore a disappointment that these rules satisfy none of the properties we formulate, as we show.

Section 2 specifies the model and establishes notation. Sections 3 and 4 are concerned with manipulation through withdrawal in economies with private endowments and in economies with a social endowment respectively. Sections 5 and 6 are concerned with manipulation through merging and splitting endowments respectively. Section 7 is concerned with manipulation through pre-delivery. Section 8 relates the current paper to existing literature and raises open questions.

2 The model

Since the types of manipulation we consider lead to variations in the population of "active" agents, allocation rules have to be defined so as to allow such variations. We model this possibility by specifying an infinite set of "potential" agents, indexed by the natural numbers $\mathbb{N}$. (In the proofs of our negative results, it suffices to consider economies with either 2 or 3 agents.) To compose an economy, a finite number of these agents are drawn from this set and their preferences specified, and resources are made available to them. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$, with generic element N. There is a finite number ℓ of goods. Each agent $i \in \mathbb{N}$ is equipped with a preference relation R_i defined on $\mathbb{R}^\ell_+$. This relation belongs to some class $\mathcal{R}$. We specify resource endowments in two alternative ways. In one case, each agent $i \in N$ is endowed with his own vector of resources $\omega_i \in \mathbb{R}^\ell_+$, his **endowment**. Given $N \in \mathcal{N}$, an **economy with agent set $\boldsymbol{N}$** is then a pair (R, ω), where $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$ is the profile of their preference relations and $\omega \equiv (\omega_i)_{i \in N} \in \mathbb{R}^{\ell N}_+$ is the profile of their individual endowments.[1] Let $\boldsymbol{\mathcal{E}^N}$ be our generic notation for a domain of economies with agent set N, and $\boldsymbol{\mathcal{E}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{E}^N$.

[1]We use the notation $\mathcal{R}^N$ to designate the cross-product of $|N|$ copies of $\mathcal{R}$ indexed by the members of N. The notation $\mathbb{R}^{\ell N}_+$ should be understood in a similar way, as the cross-product of $|N|$ copies of $\mathbb{R}^\ell_+$ indexed by the members of N.

We will be particularly interested in three domains of economies. Preferences are **classical** if they are continuous, monotone, and convex. They are **homothetic** if, whenever two bundles are indifferent, so are the two bundles obtained by multiplying them by the same positive scalar. They are **quasi-linear** if whenever two bundles are indifferent, so are the two bundles obtained by adding to each of them the same quantity of good 1. Let $\boldsymbol{\mathcal{R}_{cl}}$, $\boldsymbol{\mathcal{R}_{hom}}$, and $\boldsymbol{\mathcal{R}_{cl}}$ be the classes of classical, homothetic, and quasi-linear preferences respectively. Our first domain is the domain $\boldsymbol{\mathcal{E}_{cl}^N}$ of economies with classical preferences; our second domain is the subdomain $\boldsymbol{\mathcal{E}_{hom}^N}$ of economies in which preferences are homothetic and strictly convex; our third domain is the subdomain $\boldsymbol{\mathcal{E}_{ql}^N}$ of economies in which preferences are quasi-linear and strictly convex.

We also consider the problem of allocating a **social endowment** of resources over which all agents are understood to have equal rights. A **fair division problem with agent set $\boldsymbol{N}$** is a pair $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}^\ell_+$, where as before, R is the profile of their preference relations, and $\Omega \in \mathbb{R}^\ell_+$ is the social endowment. Let $\boldsymbol{\mathcal{F}^N}$ be our generic notation for a domain of fair division problems with agent set N, and $\boldsymbol{\mathcal{F}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{F}^N$. We define in the obvious way the domains $\boldsymbol{\mathcal{F}_{cl}^N}$, $\boldsymbol{\mathcal{F}_{hom}^N}$, and $\boldsymbol{\mathcal{F}_{ql}^N}$ of classical, homothetic, and quasi-linear fair division problems with agent set N and their extensions to arbitrary populations, $\boldsymbol{\mathcal{F}_{cl}}$, $\boldsymbol{\mathcal{F}_{hom}}$, and $\boldsymbol{\mathcal{F}_{ql}}$.

An **allocation for $\boldsymbol{(R, \omega)} \in \boldsymbol{\mathcal{E}^N}$** is a list $\{x_i\}_{i \in N} \in \mathbb{R}^{\ell N}_+$ such that $\sum x_i = \sum \omega_i$. An **allocation for $\boldsymbol{(R, \Omega)} \in \boldsymbol{\mathcal{F}^N}$** is a list $\{x_i\}_{i \in N} \in \mathbb{R}^{\ell N}_+$ such that $\sum x_i = \Omega$. Let $\boldsymbol{X(R, \omega)}$ denote the set of allocations of (R, ω) and $\boldsymbol{X(R, \Omega)}$ that of (R, Ω). A **solution defined on some domain $\boldsymbol{\mathcal{E}}$** assigns to each $N \in \mathcal{N}$ and each $e \equiv (R, \omega) \in \mathcal{E}^N$, a non-empty subset of its set of feasible allocations. A **solution defined on some domain $\boldsymbol{\mathcal{F}}$** is defined in a similar way. A **rule** is an essentially single-valued solution. Our generic notation for a rule is the letter φ.

The **Walrasian solution**, denoted $\boldsymbol{W}$, associates with each $N \in \mathcal{N}$ and each $(R, \omega) \in \mathcal{E}^N$, the set of allocations $x \in X(R, \omega)$ for which there is $p \in \Delta^{\ell-1}$ such that for each $i \in N$, $px_i \leq p\omega_i$ and for each $y_i \in \mathbb{R}^\ell_+$ with $py_i \leq p\omega_i$, $x_i \; R_i \; y_i$. To solve fair division problems, we operate this solution from equal division. We refer to this variant as the **equal-division Walrasian solution**. We denote it $\boldsymbol{W_{ed}}$. Formally, given $N \in \mathcal{N}$ and $(R, \Omega) \in \mathcal{F}^N$, $W_{ed}(R, \Omega) = W(R, (\frac{\Omega}{|N|}, \ldots, \frac{\Omega}{|N|}))$.

The **egalitarian-equivalent-in-trades solution** associates with each economy its allocations such that there is a reference trade that each agent finds indifferent to his assigned trade. The selections defined next are obtained by requiring the reference trade to lie in a pre-specified direction. To each direction is associated such a selection. Given $r \in \mathbb{R}^\ell_+$, the ***r*-egalitarian-equivalent-in-trades rule, $\boldsymbol{E^r}$,** associates with each $N \in \mathcal{N}$ and each $(R,\omega) \in \mathcal{E}^N$, the set of allocations $x \in X(R,\omega)$ for which there is $\lambda > 0$ such that for each $i \in N$, $x_i \; I_i \; \lambda r$. To simplify language, we refer to the family these rules constitute as "egalitarian".

We formulate axioms for rules that may be correspondences, and first ask whether the Walrasian solution (or the equal-division Walrasian solution) satisfies them. We prove our negative results on two subdomains where these two solutions are *single-valued*. First is the domain of economies in which preferences are homothetic and strictly convex, and individual endowments are proportional. Second is the domain of economies in which preferences are quasi-linear and strictly convex.

The proofs of the negative results are by means of two-good counterexamples. We do not provide explicit formulas for preferences, but we indicate what is needed for homotheticity or quasi-linearity of a convex preference relation. The key requirement for homotheticity in the two-good case is that if an upper-contour set at a point $a \neq 0$ has a certain line of support, then at each point above the ray from the origin passing through a, lines of support to upper contour sets should be at least as steep. For quasi-linear preferences, if an upper-contour set at a point $a \in \mathbb{R}^\ell_+$ has a certain line of support, then at each point above the horizontal line through a, lines of support to upper contour sets should be at least as steep. On either domain, for strict convexity of preferences to hold, these inequalities of slopes should be strict.

The proofs involve specifying commodity bundles and slopes of lines of support to indifference curves at these bundles, and constructing a map for which the lines of support at these points take these pre-specified values. An additional requirement may have to be satisfied that the indifference curve through one of these points should pass through some third point, or below some third point. For some proofs, several such requirements have to be met. Each proof involves showing that all the requirements can be met. In fact, in all of our examples, they can be met very simply by piece-wise linear indifference curves with at most two or three pieces. This is what we advise a reader who would be interested in constructing complete examples to do in a first step. For a more demanding reader who would prefer that preferences be smooth or strictly convex or both, approximations to piece-wise preferences with these properties can be easily defined that still respect all of the requirements.

Other notation: Let **supp(R_i, x_i)** denote the set of prices of support to agent i's upper contour set at x_i.

3 Withdrawal-proofness

First, consider an economy and suppose that an agent withdraws, taking his endowment with him; the rule is applied without him; some other agent may then be assigned a bundle such that the sum of this bundle and the endowment of the agent who withdrew can be re-divided between them so that each of them ends up with a bundle that he finds at least as desirable as his assignment if the first agent had not withdrawn, and at least one of them prefers his new bundle to his assignment then. We require of a rule that it be immune to this sort of behavior:[2]

Withdrawal-proofness: For each $(R, \omega) \in \mathcal{E}^N$, each $x \in \varphi(R, \omega)$, each $\{i, j\} \subset N$, each $x' \in \varphi(R_{N\setminus\{j\}}, \omega_{N\setminus\{j\}})$, each pair $(y_i, y_j) \in \mathbb{R}_+^{\ell\{i,j\}}$ such that $y_i + y_j = x'_i + \omega_j$, it is not the case that for each $k \in \{i, j\}$, $y_k \; R_k \; x_k$, and for at least one $k \in \{i, j\}$, $y_k \; P_k \; x_k$.

The idea can be formulated in physical terms: it should not be the case that, when an agent withdraws and the rule is applied without him, one of the agents who stay is assigned a bundle such that the sum of this bundle and the endowment of the agent who withdrew is at least as large as the sum of the bundles they would have been assigned if he had not withdrawn: using the notation of the formal statement, this version would end with: "it is not the case that $x'_i + \omega_j \geq x_i + x_j$."

[2]In the context of voting theory, a property has been proposed to prevent manipulation by withdrawal, the so-called "no-show paradox" (Brams and Fishburn, 1983). A critical difference is that that theory is concerned with the choice of a public alternative. Thus, there is no ex-post redistribution of resources. Such redistributions are essential to the manipulations we are considering here. (The fact that the set of alternatives is finite should also be noted but that is not the important difference.) The property of "pre-arranging–proofness" formulated by Sönmez (1999) in the context of several-to-one matching also implies variations in the agent set: a student and a college commit to each other prior to the operation of the matching rule and as a result ends up better off. There too, there is no counterpart to the ex-post reallocations that are essential here.

Example 1 *The Walrasian rule is not withdrawal-proof on the homothetic domain.*

Let $N \equiv \{1,2,3\}$. We construct an economy $(R,\omega) \in \mathcal{E}^N_{hom}$ in steps, as follows. First, let $\omega \equiv ((100,100),(100,100),(140,140))$. Let $x \in X(R,\omega)$ be equal to $((60,140),(60,140),(220,60))$. The trades $x_1 - \omega_1 = x_2 - \omega_2$ and $x_3 - \omega_3$ are normal to $p \equiv (1,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x = W(R,\omega)$.

Next, let agent 2 withdraw with his endowment. Let $x' \in X(R_{-2},\omega_{-2})$ be equal to $((57.5,185),(182.5,55))$. The trades $x'_1 - \omega_1$ and $x'_3 - \omega_3$ are normal to $p' \equiv (2,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x' = W(R_{-2},\omega_{-2})$.

Note that (i) $\frac{x'_{12}}{x'_{11}} \simeq 3.21 > 2.33 \simeq \frac{x_{12}}{x_{11}}$, and that (ii) $\frac{x'_{32}}{x'_{31}} \simeq .30 > .27 \simeq \frac{x_{32}}{x_{31}}$. Also, (iii) $\frac{p'_2}{p'_1} < \frac{p_2}{p_1}$. Because of (i) and (iii), we can choose $R_1 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_1, x_1)$ and $p' \in \text{supp}(R_1, x'_1)$. Because of (ii) and (iii), we can choose $R_3 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_3, x_3)$ and $p' \in \text{supp}(R_3, x'_3)$.

Now, we verify that indeed $x \in W(R,\omega)$ and $x' \in W(R_{-2},\omega_{-2})$. Finally, we observe that $x'_1 + \omega_2 = (157.5, 285) > (120, 280) = x_1 + x_2$. Thus, *withdrawal-proofness* is violated. (The property is violated in physical terms.) $\square$

Example 2 *The Walrasian rule is not withdrawal-proof on the quasi-linear domain.*

Let $N \equiv \{1,2,3\}$. We construct an economy $(R,\omega) \in \mathcal{E}^N_{ql}$ in steps, as follows. First, let $\omega \equiv ((140,140),(140,140),(100,100))$. Let $x \in X(R,\omega)$ be equal to $((160,100),(160,100),(60,180))$. The trades $x_1 - \omega_1 = x_2 - \omega_2$ and $x_3 - \omega_3$ are normal to $p \equiv (2,1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x = W(R,\omega)$.

Next, let agent 2 withdraw with his endowment. Let $x' \in X(R_{-2},\omega_{-2})$ be equal to $((200,80),(40,160))$. The trades $x'_1 - \omega_1$ and $x'_3 - \omega_3$ are normal to $p' \equiv (1,1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x' = W(R_{-2},\omega_{-2})$.

Note that (i) $x'_{12} = 80 < 100 = x_{12}$ and (ii) $x'_{32} = 160 < 180 = x_{32}$. Also, (iii) $\frac{p'_2}{p'_1} > \frac{p_2}{p_1}$. Because of (i) and (iii), we can choose $R_1 \in \mathcal{R}_{ql}$ such that $p \in \text{supp}(R_1, x_1)$ and $p' \in \text{supp}(R_1, x'_1)$. Because of (ii) and (iii), we can choose $R_3 \in \mathcal{R}_{ql}$ such that $p \in \text{supp}(R_3, x_3)$ and $p' \in \text{supp}(R_3, x'_3)$.

Now, we verify that indeed $x \in W(R,\omega)$ and $x' \in W(R_{-2},\omega_{-2})$. Finally, we observe that $x'_1 + \omega_2 = (340, 220) > (320, 200) = x_1 + x_2$. Thus, *withdrawal-proofness* is violated. (The property is violated in physical terms.) □

Example 3 *The r-egalitarian-equivalence-in-trades rules are not withdrawal-proof on the classical domain.*

In describing the example, and given two distinct points α and β, we denote by $\ell(\alpha, \beta)$ the line passing through them.

Let $N \equiv \{1, 2, 3\}$. The economy $(R, \omega) \in \mathcal{E}_{cl}^N$ is constructed in steps, as follows. Let $\omega \equiv ((160, 40), (160, 40), (140, 200))$. Let $x \in X(R, \omega)$ be equal to $((100, 100), (100, 100), (260, 80))$. The trades $x_1 - \omega_1$, $x_2 - \omega_2$, and $x_3 - \omega_3$ are normal to $p \equiv (1, 1)$. Below, we specify R so that $x \in E^r(R, \omega)$ with reference trade $t_0 = (a, a)$ for $a = 30$ (in fact, $x \in W(R, \omega)$).

Next, let agent 2 withdraw with his endowment. Let $x' \in X(R_{-2}, \omega_{-2})$ be equal to $((55, 175), (245, 65))$. The vector $\delta \equiv (b, b)$ for $b = 15$ is the vector by which agent 3's consumption will decrease as a result of this withdrawal. By feasibility, it is the vector by which, in the end, the sum of agents 1 and 2's consumptions increase. Let $x_0 \equiv \frac{\omega_3 + x_3}{2}$ and note that $\ell(x_0, x_3')$ is parallel to $\ell(x_1, x_1')$. Let $b \equiv 24$ and $c \equiv 27$. The line $\ell(x_3', \omega_3 + (b, b))$ passes below $\omega_3 + (c, c)$. Let $t_0' \equiv (c, c)$. Below, we specify R so that $x' \in E^r(R_{-2}, \omega_{-2})$ with reference trade t_0'.

We choose R_3 so that $\ell(x_3', \omega_3 + (b, b))$ is a line of support to agent 3's indifference curve through x_3' at x_3' and this indifference curve passes through $\omega_3 + t_0'$. Next, we choose agent 3's indifference curve through x_3 to be supported by $\ell(x_0, x_3)$ at x_3, to pass through $\omega_3 + t_0$, and to be compatible with the curve just drawn.

Next, we turn to agent 1. The line ℓ' parallel to $\ell(x_3', \omega_3 + (b, b))$ through x_1' passes below x_1. We specify agent 1's indifference curve through x_1' to be supported by ℓ' at x_1', to pass below x_1 and through $\omega_1 + t_0'$. We specify his indifference curve through x_1 to be supported by $\ell(\omega_1, x_1)$ at x_1, to pass through $\omega_1 + t_0$, and to be compatible with the curve just drawn.

We verify that indeed $x = E^r(R, \omega)$ with reference trade $t_0 = (a, a)$ and that $x' = E^r(R_{-2}, \omega_{-2})$. Also, because $\omega_1 + \omega_2 + \omega_3 = x_1 + x_2 + x_3$, $x_1' + x_3' = \omega_1 + \omega_3$, and $x_3' < x_3$, we have that $x_1' + \omega_2 = x_1 + x_2 + x_3 - x_3' > x_1 + x_2$. Thus *withdrawal-proofness* is violated. (The property is violated in physical terms.) □

4 Withdrawal-proofness for the problem of fair division

In the previous definition, the agent who leaves does so with his endowment. In the context of the problem of fair division, two versions of *withdrawal-proofness* can be formulated. First is a version in which the social endowment is not affected by the withdrawal. When an agent withdraws, he relinquishes his rights on the social endowment, so that the resources available for distribution among the remaining agents are what they were when he was present.

Withdrawal-proofness for fair division problems: For each $(R, \Omega) \in \mathcal{F}^N$, each $x \in \varphi(R, \Omega)$, each $\{i, j\} \subset N$, each $x' \in \varphi(R_{N\setminus\{j\}}, \Omega)$, and each pair $(y_i, y_j) \in \mathbb{R}_+^{\ell N}$ such that $y_i + y_j = x'_i$, it is not the case that for each $k \in \{i, j\}$, $y_k \; R_k \; x_k$, and for at least one $k \in \{i, j\}$, $y_k \; P_k \; x_k$.

The second scenario involves imagining that the agent who withdraws does so with an equal share of the social endowment. Then, some agent who stays may be assigned a bundle, that, when added to the bundle taken with him by the agent who withdrew, can be redivided between them so that each of them receives a bundle that he finds at least as desirable as his initial assignment and at least one of them prefers. Immunity to this kind of manipulation is a stronger requirement than *withdrawal-proofness for fair division problems*. We will not discuss it further as our results pertaining to the weak version of the property are all negative.

Withdrawal-proofness for fair division problems is related to *population monotonicity*, the requirement that as population enlarges, each of the agents initially present should end up at most as well off as he was initially (Thomson, 1983a,b bases on it characterizations of several solutions to the bargaining problem; Chichilnisky and Thomson, 1987, and Kim, 2004, are applications of the idea to standard fair division problems). Suppose that an agent withdraws from some initial economy. A violation of *withdrawal-proofness* means the following: in the resulting smaller economy, there is some other agent who, after transferring some of his assignment to the agent who withdrew, still obtains a bundle that he prefers to his assignment in the initial economy and the agent who withdrew is better off than if he had not withdrawn. If the rule is efficient, this is possible only if some third agent is worse off in the smaller economy than he was initially, in violation of *population monotonicity*.

Here too, the property can be formulated in physical terms: the formal statement would then end with "it is not the case that $x_i' \geq x_i + x_j$."

Example 4 *The equal-division Walrasian rule is not withdrawal-proof for fair division problems on the homothetic domain.*

Let $N \equiv \{1,2,3\}$. We construct an economy $(R,\Omega) \in \mathcal{E}^N_{hom}$ in steps as follows. First, let $\Omega \equiv (240,240)$. Let $x \in X(R,\Omega)$ be equal to $((60,86.66)(10,103.33),(170,50))$. The trades $x_1 - \frac{\Omega}{3} = x_2 - \frac{\Omega}{3}$ and $x_3 - \frac{\Omega}{3}$ are normal to $p \equiv (1,3)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x = W_{ed}(R,\Omega)$.

Next, let agent 2 withdraw, relinquishing his rights on the social endowment. Let $x' \in X(R_{-2},\Omega)$ be equal to $((101.25,195),(138.75,45))$. The trades $x_1' - \frac{\Omega}{2}$ and $x_3' - \frac{\Omega}{2}$ are normal to $p' \equiv (4,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x' = W_{ed}(R_{-2},\Omega)$.

Note that (i) $\frac{x_{12}'}{x_{11}'} \simeq 1.92 > 1.44 \simeq \frac{x_{12}}{x_{11}}$, and that (ii) $\frac{x_{32}'}{x_{31}'} \simeq .324 > .294 \simeq \frac{x_{32}}{x_{31}}$. Also, (iii) $\frac{p_2'}{p_1'} < \frac{p_2}{p_1}$. Because of (i) and (iii), there is $R_1 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_1,x_1)$ and $p' \in \text{supp}(R_1,x_1')$. Because of (ii) and (iii), there is $R_3 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_3,x_3)$ and $p' \in \text{supp}(R_3,x_3')$.

Now, we verify that indeed $x \in W_{ed}(R,\Omega)$ and $x' \in W_{ed}(R_{-2},\Omega)$. Finally, we observe that $x_1' = (101.25,195) > (70,190) = x_1 + x_2$. Thus, *weak withdrawal-proofness* is violated. (The property is violated in physical terms.) $\square$

Example 5 *The equal-division Walrasian rule is not withdrawal-proof for fair division problems on the quasi-linear domain.*

Let $N \equiv \{1,2,3\}$. We construct an economy $(R,\Omega) \in \mathcal{E}^N_{ql}$ in steps, as follows. First, let $\Omega \equiv (240,240)$. Let $x \in X(R,\Omega)$ be equal to $((35,95),(35,95),(170,50))$. The trades $x_1 - \frac{\Omega}{3} = x_2 - \frac{\Omega}{3}$ and $x_3 - \frac{\Omega}{3}$ are normal to $p \equiv (1,3)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x = W_{ed}(R,\Omega)$.

Next, let agent 2 withdraw, relinquishing his rights on the social endowment. Let $x' \in X(R_{-2},\Omega)$ be equal to $((104,184),(136,56))$. The trades $x_1' - \frac{\Omega}{2}$ and $x_3' - \frac{\Omega}{2}$ are normal to $p' \equiv (4,1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x' = W_{ed}(R_{-2},\Omega)$.

Note that (i) $x_{12}' = 184 > 95 = x_{12}$, and that (ii) $x_{32}' = 56 > 50 = x_{32}$. Also, (iii) $\frac{p_2'}{p_1'} < \frac{p_2}{p_1}$. Because of (i) and (iii), there is $R_1 \in \mathcal{R}_{ql}$ such that $p \in \text{supp}(R_1,x_1)$ and $p' \in \text{supp}(R_1,x_1')$. Because of (ii) and (iii), there is $R_3 \in \mathcal{R}_{ql}$ such that $p \in \text{supp}(R_3,x_3)$ and $p' \in \text{supp}(R_3,x_3')$.

Let $y_1 = y_2 \equiv \frac{x'_1}{2} = (52, 92)$. We have $py_1 = 1 \times 52 + 3 \times 92 > 1 \times 35 + 3 \times 95 = px_1$ so that we can specify R_1 in such a way that in addition $y_1 \; P_1 \; x_1$. Let $R_2 \equiv R_1$. Then, $y_2 \; P_2 \; x_2$. Thus, *weak withdrawal-proofness* is violated.□

Example 6 *The r-egalitarian-equivalence rules are not withdrawal-proof for fair division problems on the classical domain.*

Let $N \equiv \{1, 2, 3\}$. We construct an economy $(R, \Omega) \in \mathcal{E}^N_{cl}$ in steps, as follows. [Consumption bundles and endowments are as in Example 5.] First, let $\Omega \equiv (240, 240)$. Let $x \in X(R, \Omega)$ be equal to $((35, 95), (35, 95), (170, 50))$. The trades $x_1 - \frac{\Omega}{3} = x_2 - \frac{\Omega}{3}$ and $x_3 - \frac{\Omega}{3}$ are normal to $p \equiv (1, 3)$. Below, we specify $R \in \mathcal{R}^N_{cl}$ so that $x = E^r(R, \Omega)$ for $r = (1, 0)$.

Next, let agent 2 withdraw, relinquishing his rights on the social endowment. Let $x' \in X(R_{-2}, \Omega)$ be equal to $((104, 184), (136, 56))$. The trades $x'_1 - \frac{\Omega}{2}$ and $x'_3 - \frac{\Omega}{2}$ are normal to $p' \equiv (4, 1)$. Below, we specify $R \in \mathcal{R}^N_{cl}$ so that $x' = E^r(R_{-2}, \Omega)$.

Let $t_0 \equiv (70, 0)$ and $t'_0 \equiv (10, 0)$. Let $R_1 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_1, x_1)$, $x_1 \; I_1 \; (\frac{\Omega}{3} + t_0)$, $p' \in \text{supp}(R_1, x'_1)$ and $x'_1 \; I_1 \; (\frac{\Omega}{2} + t'_0)$. Also, let $R_3 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_3, x_3)$, $x_3 \; I_3 \; (\frac{\Omega}{3} + t_0)$, $p' \in \text{supp}(R_3, x'_3)$, and $x'_3 \; I_3 \; (\frac{\Omega}{2} + t'_0)$.

Let $y_1 = y_2 \equiv \frac{x'_1}{2} = (52, 92)$. We have $py_1 = 1 \times 52 + 3 \times 92 > 1 \times 35 + 3 \times 95 = px_1$ so that we can specify R_1 in such a way that in addition $y_1 \; P_1 \; x_1$. Let $R_2 \equiv R_1$. Then, $y_2 \; P_2 \; x_2$. Thus, *weak withdrawal-proofness* is violated.
□

5 Endowments-merging–proofness

Another manipulation possibility for a pair of agents is that one of them entrusts his endowment to the other and withdraws; the rule is applied without him; at the allocation that results, the second agent's assignment may be a bundle that can be divided between the two of them in such a way that each of them is at least as well off as he would have been if the merging had not taken place, and at least one of them is better off. We require immunity to this sort of behavior:

Endowments-merging–proofness: For each $(R, \omega) \in \mathcal{E}^N$, each $x \in \varphi(R, \omega)$, each $\{i, j\} \subset N$, each $x' \in \varphi(R_{N \setminus \{j\}}, \omega'_i, \omega_{N \setminus \{i,j\}})$, where $\omega'_i \equiv \omega_i + \omega_j$, and each pair $(y_i, y_j) \in \mathbb{R}^{\ell N}_+$ such that $y_i + y_j = x'_i$, it is not the case that for each $k \in \{i, j\}$, $y_k \; R_k \; x_k$, and for at least one $k \in \{i, j\}$, $y_k \; P_k \; x_k$.

The property can be formulated in physical terms: the formal statement would then end with "it is not the case that $x_i' \geq x_i + x_j$."

Example 7 *The Walrasian rule is not endowments-merging–proof on the homothetic domain.*

Let $N \equiv \{1,2,3\}$. We construct an economy $(R,\omega) \in \mathcal{E}^N_{hom}$ in steps, as follows. First, let $\omega \equiv ((60,60),(60,60),(140,140))$. Let $x \in X(R,\omega)$ be equal to $x \equiv ((35,85)(5,115),(220,60))$. The trades $x_1 - \omega_1 = x_2 - \omega_2$ and $x_3 - \omega_3$ are normal to $p \equiv (1,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x = W(R,\omega)$.

Next, let agent 2 merge his endowment with agent 1's endowment and withdraw. Let $\omega_1' \equiv \omega_1 + \omega_2$. Let $x' \in X(R_{-2},\omega_1',\omega_3)$ be equal to $((77.5,205),(182.5,55))$. The trades $x_1' - \omega_1'$ and $x_3' - \omega_3$ are normal to $p' \equiv (2,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x' = W(R_{-2},\omega_1',\omega_3)$.

Note that (i) $\frac{x'_{12}}{x'_{11}} \simeq 2.64 > 2.42 \simeq \frac{x_{12}}{x_{11}}$, and that (ii) $\frac{x'_{32}}{x'_{31}} \simeq .30 > .27 \simeq \frac{x_{32}}{x_{31}}$. Also, (iii) $\frac{p_2'}{p_1'} < \frac{p_2}{p_1}$. Because of (i) and (iii), there is $R_1 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_1, x_1)$ and $p' \in \text{supp}(R_1, x_1')$. Because of (ii) and (iii), there is $R_3 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_3, x_3)$ and $p' \in \text{supp}(R_3, x_3')$.

Now, we verify that indeed $x \in W(R,\omega)$ and $x' \in W(R_{-2},\omega_1',\omega_3)$. Finally, we observe that $x_1' = (77.5,205) > (40,200) = x_1 + x_2$. Thus, *endowments-merging–proofness* is violated. (The property is violated in physical terms.) $\square$

Example 8 *The Walrasian rule is not endowments-merging–proof on the quasi-linear domain.*

Let $N \equiv \{1,2,3\}$. We construct an economy $(R,\omega) \in \mathcal{E}^N_{ql}$ in steps, as follows. First, let $\omega \equiv ((85,50),(85,50),(140,140))$. Let $x \in X(R,\omega)$ be equal to $((70,65)(20,115),(220,60))$. The trades $x_1-\omega_1 = x_2-\omega_2$ and $x_3-\omega_3$ are normal to $p \equiv (1,1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x = W(R,\omega)$.

Next, let agent 2 merge his endowment with agent 1's endowment and withdraw. Let $\omega_1' \equiv \omega_1 + \omega_2$. Let $x' \in X(R_{-2},\omega_1',\omega_3)$ be equal to $((135,170),(175,70))$. The trades $x_1'-\omega_1'$ and $x_3'-\omega_3$ are normal to $p' \equiv (2,1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x' = W(R_{-2},\omega_1',\omega_3)$.

Note that (i) $x'_{12} = 170 > 65 = x_{12}$, and that (ii) $x'_{32} = 70 > 60 = x_{32}$. Also, (iii) $\frac{p_2'}{p_1'} < \frac{p_2}{p_1}$. Because of (i) and (iii), there is $R_1 \in \mathcal{R}_{ql}$ such that

$p \in \mathrm{supp}(R_1, x_1)$ and $p' \in \mathrm{supp}(R_1, x_1')$. Because of (ii) and (iii), there is $R_3 \in \mathcal{R}_{ql}$ such that $p \in \mathrm{supp}(R_3, x_3)$ and $p' \in \mathrm{supp}(R_3, x_3')$.

Now, we verify that indeed $x \in W(R, \omega)$ and $x' \in W(R_{-2}, \omega_1', \omega_3)$. Let $y_1 = y_2 \equiv \frac{x_1'}{2} = (67.5, 85)$. Because $py_1 = 67.5 + 85 > 70 + 65 = px_1$, then R_1 can be specified so that in addition $y_1\ P_1\ x_1$. Because $R_2 = R_1$, then $y_2\ P_2\ x_2$. Thus, *endowments-merging–proofness* is violated. □

Example 9 *The egalitarian-equivalence-in-trades rules are not endowments-merging–proof on the classical domain.*

Let $N \equiv \{1, 2, 3\}$. We construct an economy $(R, \omega) \in \mathcal{E}_{ql}^N$ in steps, as follows. First, let $\omega \equiv ((80, 80), (80, 80), (220, 140))$. Let $x \in X(R, \omega)$ be equal to $((40, 120)(40, 120), (300, 60))$. The trades $x_1 - \omega_1 = x_2 - \omega_2$ and $x_3 - \omega_3$ are normal to $p \equiv (1, 1)$. Below, we specify $R \in \mathcal{R}_{cl}^N$ so that $x = E^r(R, \omega)$ for $r \equiv (1, 1)$.

Next, let agent 2 merge his endowment with agent 1's endowment and withdraw. Let $\omega_1' \equiv \omega_1 + \omega_2$. Let $x' \in X(R_{-2}, \omega_1', \omega_3)$ be equal to $((100, 260), (280, 40))$. The trades $x_1' - \omega_1'$ and $x_3' - \omega_3$ are normal to $p' \equiv (5, 3)$. Below, we specify $R \in \mathcal{R}_{cl}^N$ so that $x' = E^r(R_{-2}, \omega_1', \omega_3)$.

Let $t_0 \equiv (20, 20)$ and $t_0' \equiv (10, 10)$. There is $R_1 \in \mathcal{R}_{cl}$ such that $p \in \mathrm{supp}(R_1, x_1)$, $p' \in \mathrm{supp}(R_1, x_1')$, $x_1\ I_1\ (\omega_1 + t_0)$ and $x_1'\ I_1\ (\omega_1' + t_0')$. Because $R_2 = R_1$, then $p \in \mathrm{supp}(R_2, x_2)$, and $x_2\ I_2\ (\omega_2 + t_0)$. Also, there is $R_3 \in \mathcal{R}_{cl}$ such that $p \in \mathrm{supp}(R_3, x_3)$, $p' \in \mathrm{supp}(R_3, x_3')$, $x_3\ I_3\ (\omega_3 + t_0)$ and $x_3'\ I_3\ (\omega_3 + t_0')$.

Now, we verify that indeed $x = E^r(R, \omega)$ and $x' = E^r(R_{-2}, \omega_1', \omega_3)$. Finally, we note that $x_1' = (100, 260) > (80, 240) = x_1 + x_2$. Thus, *endowments-merging–proofness* is violated.(The violation is in physical terms.) □

6 Endowments-splitting–proofness

Symmetrically to the behavior examined in the previous section, here an agent transfers some of his endowment to some agent who was not initially present; the rule is applied, and the guest transfers his assignment to the agent who invited him; now the first agent may have access to a bundle that he prefers to his initial assignment. We require immunity to this sort of behavior:

Endowments-splitting–proofness: For each $(R,\omega) \in \mathcal{E}^N$, each $x \in \varphi(R,\omega)$, each $i \in N$, each $j \notin N$, each $R'_j \in \mathcal{R}$, and each pair $(\omega'_i, \omega'_j) \in \mathbb{R}_+^{\ell\{i,j\}}$ such that $\omega'_i + \omega'_j = \omega_i$, each $x' \in \varphi(R, R'_j, \omega'_i, \omega_{N\setminus\{i\}}, \omega'_j)$, we have $x_i \; R_i \; (x'_i + x'_j)$.

The possibility that agents in a group may gain by transferring endowments among themselves has been considered previously (Gale, 1974; Postlewaite, 1979). In the context of the adjudication of conflicting claims, manipulation takes the form of claims transfers. By contrast to the phenomenon studied here, these manipulations do not affect the set of active agents. However, the merging and splitting of claims that has played an important role in the study of the adjudication of conflicting claims does (O'Neill, 1982; de Frutos, 1999; Ju, 2003; Ju, Miyagawa, and Sakai, 2007; Ju and Moreno-Ternero, 2006, 2011), and so does the merging and splitting of time allotments in the theory of scheduling (Moulin, 2007, 2008). Returning to our model, it is clear that the manipulation will be made easier if no restrictions are imposed on the preferences of the new agent. A natural such restriction is that the new agent's preferences are the same as those of some agent initially present. We operate under that constraint in constructing our examples.

The property can be formulated in physical terms, substituting in the conclusion of our earlier statement the statement "then $x_i \geqq (x'_i + x'_j)$."[3]

Example 10 *The Walrasian rule is not endowments-splitting–proof on the homothetic domain.*

Let $N \equiv \{1,3\}$. We construct an economy $(R,\omega) \in \mathcal{E}^N_{hom}$ in steps, as follows. First, let $\omega \equiv ((120,120),(120,120))$. Let $x \in X(R,\omega)$ be equal to $x \equiv ((60,180)(180,60))$. The trades $x_1 - \omega_1$ and $x_3 - \omega_3$ are normal to $p \equiv (1,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x = W(R,\omega)$.

Next, let agent 1 split his endowment with a new agent, agent 2, whose preferences R_2 are the same as those of agent 3. Let $\omega'_1 = \omega'_2 \equiv \frac{\omega_1}{2}$. Let $x' \in X(R_1,R_2,R_3,\omega'_1,\omega'_2,\omega_3)$ be equal to $((27,159)(71,27),(142,54))$. The trades $x'_1 - \omega'_1$, $x'_2 - \omega'_2$, and $x'_3 - \omega_3$ are normal to $p' \equiv (3,1)$. Below, we specify $R \in \mathcal{R}^N_{hom}$ so that $x' = W(R_1,R_2,R_3,\omega'_1,\omega'_2,\omega_3)$.

Note that (i) $\frac{x'_{12}}{x'_{11}} \simeq 5.88 > .30 = \frac{x_{12}}{x_{11}}$ and that (ii) $\frac{x'_{32}}{x'_{31}} \simeq .38 > .33 \simeq \frac{x_{32}}{x_{31}}$. Also, (iii) $\frac{p'_2}{p'_1} < \frac{p_2}{p_1}$. Because of (i) and (iii), there is $R_1 \in \mathcal{R}_{hom}$ such that

[3] Vector inequalities: $x \geqq y$; $x \geq y$, $x > y$.

$p \in \text{supp}(R_1, x_1)$ and $p' \in \text{supp}(R_1, x'_1)$. Because of (ii) and (iii), there is $R_3 \in \mathcal{R}_{hom}$ such that $p \in \text{supp}(R_3, x_3)$ and $p' \in \text{supp}(R_3, x'_3)$. Also, because $\frac{x'_{22}}{x'_{21}} = \frac{x'_{32}}{x'_{31}}$, we can choose $R_2 = R_3$.

Now, we verify that indeed $x \in W(R, \omega)$ and $x' \in W(R_1, R_2, R_3, \omega'_1, \omega'_2, \omega_3)$. Note that $x'_1 + x'_2 = (98, 186) > (60, 180) = x_1$. Thus, *endowments-splitting–proofness* is violated in physical terms. □

Example 11 *The Walrasian rule is not endowments-splitting–proof on the quasi-linear domain.*

Let $N \equiv \{1, 3\}$. We construct an economy $(R, \omega) \in \mathcal{E}^N_{ql}$ in steps, as follows. First, let $\omega \equiv ((40, 180)(200, 60))$. Let $x \in X(R, \omega)$ be equal to $x \equiv ((80, 100)(160, 140))$. The trades $x_1 - \omega_1$ and $x_3 - \omega_3$ are normal to $p \equiv (2, 1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x = W(R, \omega)$.

Next, let agent 1 split his endowment with a new agent, agent 2, whose preferences R_2 are the same as his. Let $\omega'_1 = \omega'_2 \equiv \frac{\omega_1}{2}$. Let $x' \in X(R_1, R_2, R_3, \omega'_1, \omega'_2, \omega_3)$ be equal to $((50, 60), (50, 60), (140, 120))$. The trades $x'_1 - \omega'_1$, $x'_2 - \omega'_2$, and $x'_3 - \omega_3$ are normal to $p' \equiv (1, 1)$. Below, we specify $R \in \mathcal{R}^N_{ql}$ so that $x' = W(R_1, R_2, R_3, \omega'_1, \omega'_2, \omega_3)$.

Note that (i) $x_{12} = 100 > 60 = x'_{12}$, and that (ii) $x_{32} = 140 > 120 = x'_{32}$. Also, (iii) $\frac{p'_2}{p'_1} > \frac{p_2}{p_1}$. Because of (i) and (iii), there is $R_1 \in \mathcal{R}_{ql}$ such that $p \in \text{supp}(R_1, x_1)$ and $p' \in \text{supp}(R_1, x'_1)$. Because of (ii) and (iii), there is $R_3 \in \mathcal{R}_{ql}$ such that $p \in \text{supp}(R_3, x_3)$ and $p' \in \text{supp}(R_3, x'_3)$.

Now, we verify that indeed $x \in W(R, \omega)$ and $x' \in W(R_1, R_2, R_3, \omega'_1, \omega'_2, \omega_3)$. Note that $x'_1 + x'_2 = (100, 120) > (80, 100) = x_1$. Thus, *endowments-splitting–proofness* is violated. (The violation is in in physical terms.) □

Example 12 *The r-egalitarian-equivalence-in-trades rules are not endowments-splitting–proof on the classical domain.*

Let $N \equiv \{1, 3\}$. We construct an economy $(R, \omega) \in \mathcal{E}^N_{cl}$ in steps, as follows. First, let $\omega \equiv ((160, 160), (200, 140))$. Let $x \in X(R, \omega)$ be equal to $x \equiv ((80, 240), (280, 60))$. The trades $x_1 - \omega_1$ and $x_3 - \omega_3$ are normal to $p \equiv (1, 1)$. Let $r \equiv (1, 1)$. Below, we specify $R \in \mathcal{R}^N_{cl}$ so that $x = E^r(R, \omega)$ (in fact, $x = W(R, \omega)$).

Next, let agent 1 split his endowment with a new agent, agent 2, whose preferences R_2 are the same as his. Let $\omega'_1 = \omega'_2 \equiv \frac{\omega_1}{2}$. Let

$x' \in X(R_1, R_2, R_3, \omega_1', \omega_2', \omega_3')$ be equal to $((50, 130), (50, 130), (260, 40))$. The trades $x_1' - \omega_1'$, $x_2' - \omega_2'$, and $x_3' - \omega_3$ are normal to $p' \equiv (5, 3)$. Below, we specify $R \in \mathcal{R}_{cl}^N$ so that $x' = E^r(R_1, R_2, R_3, \omega_1', \omega_2', \omega_3)$ (in fact, $x' = W(R_1, R_2, R_3, \omega_1', \omega_2', \omega_3)$).

Let $t_0 \equiv (20, 20)$ and $t_0' \equiv (10, 10)$. We note that there is $R_1 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_1, x_1)$, $p' \in \text{supp}(R_1, x_1')$, $x_1\ I_1\ (\omega_1 + t_0)$, and $x_1'\ I_1\ (\omega_1' + t_0')$. Also, there is $R_3 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_3, x_3)$, $p' \in \text{supp}(R_3, x_3')$, $x_3\ I_3\ (\omega_3 + t_0)$, and $x_3'\ I_3\ (\omega_3' + t_0')$.

Now, we verify that indeed $x \in E^r(R, \omega)$ and $x' \in E^r(R_1, R_2, R_3, \omega_1', \omega_2', \omega_3)$. Note that $x_1' + x_2' = (100, 260) > (80, 240) = x_1$. Thus, *endowments-splitting–proofness* is violated.(The violation is in physical terms.) $\square$

7 Pre-delivery–proofness

Our final scenario involves one agent making a "pre-delivery" to some other agent of the trade that this other agent would be assigned if he, the second agent, had participated with everyone else. The agent who undertakes the pre-delivery starts out with a different endowment. After the rule is applied, he may end up with a bundle that he prefers to his assignment if he had not carried out the pre-delivery. We require immunity to this sort of behavior:

Pre-delivery–proofness: For each $(R, \omega) \in \mathcal{E}^N$, each $x \in \varphi(R, \omega)$, each $\{i, j\} \subset N$ such that $\omega_i + \omega_j - x_j \geqq 0$, each $x' \in \varphi(R_{N\setminus\{j\}}, \omega_i', \omega_{N\setminus\{i,j\}})$ where $\omega_i' \equiv \omega_i + \omega_j - x_j$, we have $x_i\ R_i\ x_i'$.

The property can also be formulated for the problem of fair division.

We have a negative result for the egalitarian rules:

Example 13 *The r-egalitarian-equivalence-in-trades rules is not pre-delivery–proof on the classical domain.*

Let $N \equiv \{1, 2, 3\}$. We construct an economy $(R, \omega) \in \mathcal{E}_{cl}^N$ in steps, as follows. First, let $\omega \equiv ((120, 80), (120, 80), (140, 200))$. Let $x \in X(R, \omega)$ be equal to $((40, 160), (160, 40), (180, 160))$. The trades $x_1 - \omega_1$ and $x_3 - \omega_3$ are normal to $p \equiv (1, 1)$. Below, we specify $R \in \mathcal{R}_{cl}^N$ so that $x = E^r(R, \omega)$ with $r = (1, 1)$.

We then let agent 1 pre-delivers his net trade to agent 2. His revised endowment is $\omega_1' \equiv \omega_1 - x_2 + \omega_2 = (80, 120)$. Let $x' \in X(R_1, R_3, \omega_1', \omega_3)$ be equal to $((50, 170)), (170, 150))$. Below, we specify $R \in \mathcal{R}_{cl}^N$ so that $p \in \text{supp}(R_1, x_1')$ and $p \in \text{supp}(R_3, x_3')$.

Let $t_0 \equiv (20, 20)$ and $t_0' \equiv (15, 15)$. There is $R_1 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_1, x_1)$, $p \in \text{supp}(R_1, x_1')$, $x_1\ I_1\ (\omega_1 + t_0)$, and $x_1'\ I_1\ (\omega_1' + t_0')$. There is $R_2 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_2, x_2)$ and $x_2\ I_2\ (\omega_2 + t_0)$. Finally, there is $R_3 \in \mathcal{R}_{cl}$ such that $p \in \text{supp}(R_3, x_3)$, $x_3\ I_3\ (\omega_3 + t_0)$, $p \in \text{supp}(R_3, x_3')$, and $x_3'\ I_3\ (\omega_3 + t_0')$.

Now, we verify that indeed $x \in E^r(R, \omega)$ with $r \equiv (1, 1)$, and that $x' \in E^r(R_1, R_3, \omega_1', \omega_3)$. Finally, we observe that $x_1' = (50, 170) > (40, 160) = x_1'$. Thus, *pre-delivery–proofness* is violated. (The violation is in physical terms.) $\square$

It is easy to see that the same negative result holds when the endowments are equal initially.

The Walrasian rule is *pre-delivery–proof.* Indeed, if endowments change in the manner described in the hypotheses of the axiom, their values at the initial equilibrium prices remain the same, so that these prices remain equilibrium prices. (In fact, consider any balanced reassignment of endowments such that each agent's revised endowment remains in his initial equilibrium hyperplane. Then, if the same prices are quoted, the same list of bundles for the agents who stay still satisfy the Walrasian maximization requirements.) The equal-division Walrasian rule is also obviously *pre-delivery–proof.*

Thus, we end with the following positive results.

Proposition 1 *The Walrasian rule is pre-delivery–proof, and so is the equal-division Walrasian rule.*

8 Concluding comments

We conclude with some remarks and open questions.

1. In light of the mainly negative results we formulated, it is natural to explore variants of the requirements obtained by restricting how agents can manipulate. The manipulation is less likely to be profitable if one insists that the manipulating agents (agents i and j in our original definitions) be able to divide the resources they end up controlling in such a way that:

(i) A simple redistribution scheme is used in the second stage. In that respect, we have already noted that in most of the examples we construct for our proofs, the welfare improvements for the two agents engaging in the manipulation is obtained by simply dividing equally the resources they end up controlling. Thus, at that stage at least, and for the examples, no sophisticated redistribution scheme is needed. Equal division is not efficient for them, so by trading from there, they can achieve further welfare gains, providing them with an additional incentive to manipulate.

(ii) The same rule is used in both rounds. This is a probably less natural restriction. If agents have the ability to escape detection in the manner in which they manipulate endowments, it is not clear why they should be compelled to respect the rule when distributing the gains they can achieve by this behavior.

2. Other forms of manipulation of allocation rules through manipulations of endowments have been studied, and for some of them, an extensive literature is available. A well-known possibility is that an agent may gain by transferring some of his endowment to another agent, without the other agent being hurt (Gale, 1974). More generally, one may be concerned about arbitrary transfers of endowments among several agents (Postlewaite, 1979). An agent may also gain by withholding some of his endowment (Postlewaite, 1979; Thomson, 1987). An agent may even gain by destroying some of his endowment (Aumann and Peleg, 1974; Postlewaite, 1979). Finally, an agent may gain by artificially augmenting his endowment through exaggeration or borrowing (Thomson, 2005). What distinguishes these forms of manipulation from the ones we formulated here is that none of them is accompanied by variations in the set of active agents. Our focus here was on that possibility.

3. For each of the behaviors we have discussed, we could imagine more than two agents participating. For instance, a group of agents could withdraw with their endowments and after the rule is applied to the smaller economy involving the remaining agents, look for a redistribution of the resources that they and several of the agents who stayed now control, at which each of these conspirators would be at least as well off and at least one of them better off. Our negative results are made all the stronger by the fact that we considered minimal groups for which the manipulation is meaningful.

On the other hand, the positive result concerning the *pre-delivery–proofness* of the Walrasian rule is not robust to the formation of larger groups of manipulating agents. Indeed, when a group of two agents, say, get together to pre-deliver to some third agent the net trade that this agent has been assigned by this rule in some initial economy, they may do so in such a way that their revised endowments do not belong to their original budget hyperplanes. The situation then is akin to one mentioned in the preceding paragraph, when in a fixed-population model, two agents manipulate by redistributing endowments among themselves before participating. It is not exactly the same because here the manipulation opportunities open to the conspirators are constrained by the fact that the sum of their revised endowments will differ from its pre-manipulation value (the difference is the trade assigned to the third agent), whereas in the fixed-population version, the pre-manipulation and post-manipulation sums are equal. Still, in the new economy, it is clear that there is no reason why the original equilibrium prices should remain so, and easy to imagine that the conspirators could gain.

Sertel and Yıldız (2004) consider a scenario according to which the agents who leave apply the rule among themselves and when the conspirators get together, the rule is applied again. They ask whether the core can be manipulated in this manner.

4. Going beyond the two specific rules that we have examined, we would of course like to know whether the principles we have formulated are compatible with efficiency and any of the distributional concepts that have been studied in the literature (the endowments lower bound, no-envy-in-trades, egalitarian-equivalence-in-trades). We leave these questions to future research. In the light of what we know about related properties, we suspect that the answers will be negative.[4]

[4]Consider rules defined as follows: a reference order is chosen on the set of potential agents; for each economy, we select the efficient allocation(s) at which the agent who is first in the order induced by the reference order on the population of agents who are actually present reaps all the gains from exchange subject to each of the others being assigned a bundle that he finds at least as desirable as his endowment. A small step in the direction of proving our conjectures is that these constrained sequential priority rules (constrained by the endowments lower bound requirement) satisfy none of the properties we formulate. The proofs of these negative results, which hold on both the homothetic domain and the quasi-linear domain, can be obtained from the author upon request.

REFERENCES

Aumann, R. and B. Peleg, "A note on Gale's example," *Journal of Mathematical Economics* 1 (1974) 209-211.

Brams, S. and P. Fishburn, "Paradoxes of preferential voting", *Mathematics Magazine* 56 (1983), 207-214.

Chichilnisky, G. and W. Thomson, "The Walrasian mechanism from equal division is not monotonic with respect to variations in the number of consumers," *Journal of Public Economics* 32 (1987), 119-124.

Foley, D., "Resource allocation and the public sector", *Yale Economic Essays* 7 (1967), 45-98.

de Frutos, M. A. "Coalitional manipulation in a bankruptcy problem", *Review of Economic Design* 4 (1999), 255-272.

Gale, D., "Exchange equilibrium and coalitions", *Journal of Mathematical Economics* 1 (1974), 63-66.

Ju, B.-G., "Manipulation via merging and splitting in claims problems", *Review of Economic Design* 8 (2003), 205-215.

Ju, B.-G., E. Miyagawa, and T. Sakai, "Non-manipulable division rules in claims problems and generalizations", *Journal of Economic Theory* 127 (2007), 1-28.

Ju, B.-G. and J. Moreno-Ternero, "Progressivitity, inequality reduction and merging-proofness in taxation", mimeo, 2006.

—— and ——, "Progressive and merging-proof taxation", *International Journal of Game Theory* 40 (2011), 43-62.

Kim, H., "Population monotonicity for fair allocation problems", *Social Choice and Welfare* 23 (2004), 59-70.

Moulin, H., "On scheduling fees to prevent merging, splitting and transferring of jobs", *Mathematics of Operations Research* 32 (2007), 266-283.

——, "Proportional scheduling, split-proofness, and merge-proofness", *Games and Economic Behavior* 63 (2008), 567-587.

O'Neill, B., "A problem of rights arbitration from the Talmud", *Mathematical Social Sciences* 2 (1982), 345-371.

Pazner, E. and D. Schmeidler, "Egalitarian equivalent allocations: A new concept of economic equity", *Quarterly Journal of Economics* 92 (1978), 671-687.

Postlewaite, A., "Manipulation via endowments", *Review of Economic Studies* 46 (1979), 255-262.

Schmeidler, D. and K. Vind, "Fair net trades", *Econometrica* 40 (1972), 637-42.

Sertel, M. and M. Yıldız, “The core is manipulable by segmentation”, *Journal of Economic Theory* 118 (2004), 103-117.

Sönmez, T., “Can pre-arranged matches be avoided in two-sided matching markets?”, *Journal of Economic Theory* 86 (1999), 148-156.

Thomson, W., “The fair division of a fixed supply among a growing population”, *Mathematics of Operations Research* 8 (1983a), 319-326.

——, “Problems of fair division and the egalitarian principle,” *Journal of Economic Theory* 31 (1983b), 211-226.

——, “Monotonic allocation rules,” University of Rochester Discussion Paper No. 116, December 1987, revised 1995.

——, “Borrowing-proofness”, mimeo, June 2005.