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Equal area rule to adjudicate conflicting claims

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Abstract

We consider the adjudication of conflicting claims over a resource. By mapping such a problem into a bargaining problem à la Nash, we avail ourselves of the solution concepts developed in bargaining theory. We focus on the solution to two-player bargaining problems known as the "equal area solution". We study the properties of the induced rule to solve claims problems. We identify difficulties in extending it from two claimants to more than two claimants, and propose a resolution.

JEL Classification numbers: C79; D63; D74.
Key-words: claims problem; equal area solution; consistency.

*University of Rochester, Department of Economics, Rochester, NY 14627, USA. In homage to Leo Hurwicz, who laid the foundations of the theory of economic design. I thank Patrick Harless for his extensive comments. Equalarearule.tex

1 Introduction

When a firm goes bankrupt, how should its liquidation value be divided among its creditors? A "rule" is a mapping that specifies, for each situation of this kind, which we call a "claims problem", a division of this value. Alternatively, the problem may be that of specifying the contributions that a group of taxpayers should make to the cost of a public project as a function of their incomes. The formal literature on the subject, whose goal is to identify the most desirable rules, originates in O'Neill (1982).[1]

In the search for rules to solve any type of resource allocation problems, it is a common strategy to invoke concepts from the theory of cooperative games, bargaining games or coalitional-form games. The allocation problems under consideration are mapped into games, a solution defined on the class of games to which these games belong is applied, and the allocations whose images are the resulting payoff vectors are selected for the allocation problems.

For claims problems, this strategy has been followed by Dagan and Volij (1993), who proposed a simple and natural way of mapping claims problems into bargaining games (Nash, 1950), and then focused on commonly used solutions to the bargaining problem, the Nash solution and its weighted versions, and the Kalai-Smorodinsky solution. Other solutions have been defined for bargaining games that are based on measuring in some fashion the sacrifice imposed on each player at a proposed payoff vector, and in selecting a vector at which sacrifices are equal across players. The "equal area" solution is a two-player solution of this type. Given a game, a player's sacrifice at a payoff vector is simply measured by the area of the set of feasible vectors at which his payoff is larger (Anbarci, 1993; Anbarci and Bigelow, 1994; Calvo and Peters, 2000; Thomson, 1996).[2] As an argument why they are not getting enough at a proposed compromise, people often point to the alternatives at which they could get more, how numerous these alternatives are, how far the compromise would place them from their most preferred alternative, as compared to how others would be treated according to such criteria.

The equal area solution is not as central in the theory of bargaining, but it enjoys a number of appealing properties. In particular, being quite

[1]For surveys, see Thomson (2003, 2015, 2018).

[2]A family of rules are introduced by Young (1987) under the name of "equal sacrifice" rules". Our solution is not a member of this family.

sensitive to the shape of the feasible set, it does not suffer from the occasional paradoxical behaviors of other rules. This sensitivity is a disadvantage in other respects: applying the equal area solution requires the knowledge of the entire feasible set. By the same token, it prevents the solution from satisfying certain invariance properties that one may be interested in. Thus, the rule provides another illustration of the familiar tradeoff in the design of allocation rules between sensitivity and simplicity.

Here, following Ortells and Santos (2011), we apply the equal area solution to solve two-claimant claims problems, obtaining a rule we call the equal area rule. The complexity issue just discussed does not arise in the context of claims problems because the boundary of the feasible set is linear and in fact, an explicit algebraic formula can be given for the equal area rule.[3] Dagan and Volij's choice of the Nash and Kalai-Smorodinsky solutions led them to well-known rules for claims problems, but the equal area rule is new. We begin by studying its properties.

We find that it satisfies all of the basic properties that have been formulated in the literature on claims problems, including all monotonicity properties. The properties that it does not satisfy are mainly invariance properties, which should not be surprising, in the light of our earlier comments on its sensitivity to the shape of the feasible set. One property of that type that it does satisfy however is invariance with respect to truncation of claims at the endowment.[4]

We then turn to problems with more than two claimants. There is more than one way of generalizing the two-claimant equal area bargaining solution to arbitrarily many players, and we briefly discuss the reasons why. These difficulties apply here as well. In the face of this multiplicity, we invoke an important property of allocation rules, called consistency, which has successfully guided the search for extensions of two-agent rules in a great variety of contexts. For claims problems, its expression is particularly simple: a rule is consistent if for each problem, the awards vector it selects is such that for each subgroup of claimants, it selects the restriction of that vector to this population for the problem of allocating among them the amount that

[3] An application of the idea to classical fair allocation problems is proposed and studied by Velez and Thomson (2012).

[4] Incidentally, this property is necessary and sufficient condition for a rule to be obtainable as the composition of two mappings: one is O'Neill's mapping from claims problems to transferable utility coalitional games; the other is a solution for this class of games (Curiel, Maschler and Tijs, 1987).

remains available after the other claimants have collected their awards and left. Unfortunately, as we show, the two-claimant equal area rule has no consistent extension. In the light of this negative result, we turn to the weaker notion of average consistency (Dagan and Volij, 1997), which still captures much of what consistency itself conveys. This notion allows an extension, and this extension is unique. We discuss some of its properties.

2 The model and the equal area rule

A group of **agents**, N, have **claims**, $(c_i)_{i\in N}$, on an infinitely divisible resource. These claims add up to more than what is available, the **endowment**, E. Thus, a **claims problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i\in N} c_i \geq E$.[5] Let $\mathcal{C}^N$ denote the class of all claims problems.

An **awards vector** for (c, E) is a vector $x \in \mathbb{R}^N_+$ satisfying the **non-negativity** and **claims boundedness** inequalities $0 \leqq x \leqq c$ and the **balance** equality $\sum x_i = E$. We refer to the line of equation $\sum x_i = E$ as a **budget line**. A **rule** is a mapping that associates with each problem in $\mathcal{C}^N$ an awards vector for it. The **path of awards of a rule** S for a claims vector $c \in \mathbb{R}^N_+$ is the locus of the awards vector S selects for (c, E) as E ranges from 0 to $\sum c_i$. We denote it $p^S(c)$.

For our purposes, it will suffice to define a **bargaining game** with player set N (Nash, 1950) as a convex, compact, and comprehensive[6] subset of $\mathbb{R}^N_+$ that contains at least one point whose coordinates are all positive.[7] A **bargaining solution** associates with each such game a point of it. Let $\mathcal{B}^N$ be the class of all bargaining games.

The bargaining solution that is our point of departure is defined for two players. Let $N \equiv \{1, 2\}$. Given $S \in \mathcal{B}^N$, the **equal area solution**, A, selects the undominated point of S with the property that the area $\alpha_1(S, x)$ of the set of points of S of abscissa greater than x_1 is equal to the area $\alpha_2(S, x)$ of

[5] We denote by $\mathbb{R}^N_+$ the cartesian product of $|N|$ copies of $\mathbb{R}_+$ indexed by the members of N. The superscript N may also indicate some object pertaining to the set N. Which interpretation is the right one should be clear from the context. We allow the equality $\sum_{i\in N} c_i = E$ for convenience.

[6] A subset S of $\mathbb{R}^N_+$ is comprehensive if for each $x \in S$ and each $0 \leq y \leq x$, $y \in S$.

[7] The usual specification of a bargaining game includes a disagreement point, and our formulation amounts to assuming that it is the origin. This assumption is justified if the theory is required to be independent of the choice of origin for the utility functions that are used to represent the opportunities available to the agents.

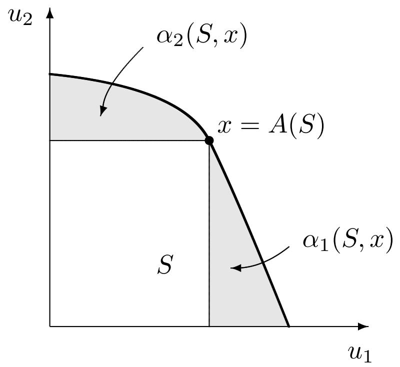

Figure 1: For two players, the equal area solution. The equal area solution selects the undominated point x of S at which the two curvi-linear triangles defined by the boundary of S and lines parallel to the axes through x have equal areas: $\alpha_1(S,x) = \alpha_2(S,x)$.

the set of points of S of ordinate greater than x_2 (Figure 1).

Given a claims problem $(c, E) \in \mathcal{C}^N$, its associated bargaining game $B(c, E)$ consists of the points of $\mathbb{R}^N_+$ that are dominated by c and lie below the budget line. The equal area bargaining solution leads directly to the following rule for claims problems (Ortells and Santos, 2011):

Equal area rule, A: Let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$. Then, $A(c, E)$ is the awards vector x with the property that among the non-negative vectors that are dominated by c and lie below the budget line, the area of the region to the right of the vertical line through x is equal to the area of region above the horizontal line through x.

Other notation. Given $a, b, c \in \mathbb{R}^N$, $\Delta(a, b, c)$ denotes the triangle with vertices a, b, and c.

Because the bargaining problem associated with a claims problem is a rectangle truncated by a line of slope -1, the coordinates of its equal area point can be calculated explicitly. They are given in the following lemma. Let $c \in \mathbb{R}^N_+$. The lemma says that $p^A(c)$ has three parts, corresponding to a three-way partition of the set of possible values of the endowment given c. They are represented in the three panels of Figure 2.

Lemma 1 (Ortells and Santos, 2011)**.** *Let $N \equiv \{1, 2\}$ and $(c, E) \in \mathcal{C}^N$ be such that $c_1 < c_2$, say. The coordinates of its equal area awards vector are as follows:*

Case 1: $E \leq c_1$: $A(c, E) = (\frac{E}{2}, \frac{E}{2})$.

Case 2: $E \in [c_1, c_2]$: $A(c, E) = (c_1[1 - \frac{c_1}{2E}], E - c_1[1 - \frac{c_1}{2E}])$.

Case 3: $E \geq c_2$: $A(c, E) = (\frac{E}{2} + (c_1 - c_2)(1 - \frac{c_1+c_2}{2E}), E - \frac{E}{2} - (c_1 - c_2)(1 - \frac{c_1+c_2}{2E}))$.

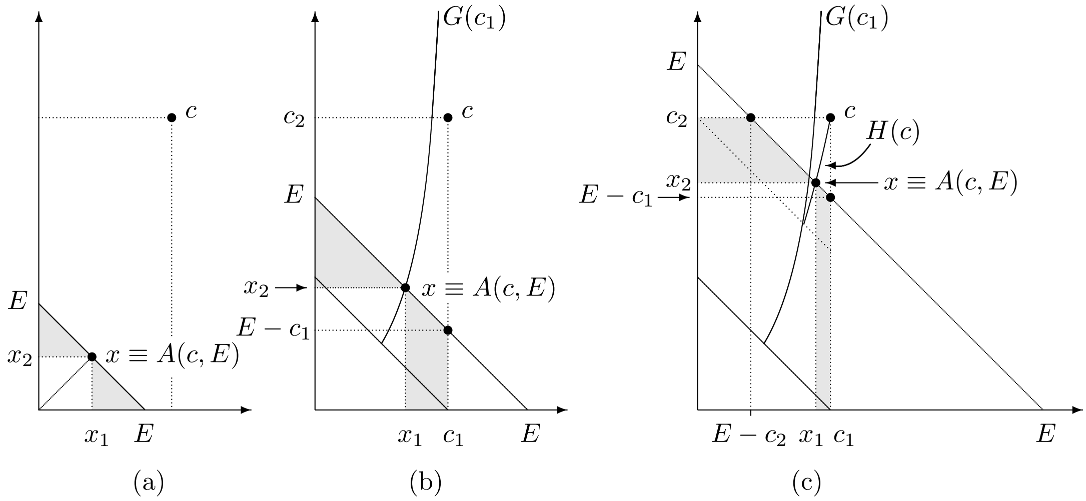

Figure 2: Constructing a path of awards of the equal area rule. Here, $c_1 < c_2$. The path has three parts, each corresponding to one of the three intervals into which the range of variations in the endowment can be partitioned. (a) When $E \leq c_1$. (b) When $c_1 \leq E \leq c_2$. (c) When $E \geq c_2$.

In each of the three cases enumerated in the lemma, the coordinates of $A(c,E)$ are obtained by writing equality of

Case 1: the area of $\Delta(x,(x_1,0),(E,0))$ and the area of $\Delta(x,(0,x_2),(0,E))$ (panel (a)).

Case 2: the difference between the areas of $\Delta(x,(x_1,0),(E,0))$ and $\Delta((c_1,E-c_1),(c_1,0),(E,0))$, and the area of $\Delta(x,(0,x_2),(0,E))$ (panel (b)).

Case 3: the difference between the areas of $\Delta(x,(x_1,0),(E,0))$ and $\Delta((c_1,E-c_1),(c_1,0),(E,0))$, and the difference between the areas of $\Delta(x,(0,x_2),(0,E))$ and $\Delta((E-c_2,c_2),(0,c_2),(0,E))$, (panel (c)).

In Case 2, the coordinates of $A(c,E)$ do not depend on c_2. Given $c_0 \in \mathbb{R}_+$, let $G(c_0)$ be the locus of the point $(c_0[1-\frac{c_0}{2E}], E-c_0[1-\frac{c_0}{2E}])$ as E varies in $[c_0,\infty[$. Later on, we will consider claims vectors for the group $\{1,3\}$ in which agent 1's claim is the smaller one, and for the group $\{2,3\}$ in which agent 2's claim is the smaller one, and we will construct the paths of awards of the equal area rule for these claims vectors. Then, the notation $G(c_1)$ and $G(_2)$ will designate the copy of the curve we just defined in the spaces $\mathbb{R}^{\{1,3\}}$ and $\mathbb{R}^{\{2,3\}}$. For $p^A(c)$, we only need the part of it that corresponds to E varying in $[\min c_i, \max c_i]$.

In Case 3, the locus of $A(c,E)$ as E varies in $[\max c_i, c_1+c_2]$ is a curve that we call $H(c)$.

3 Properties of the equal area rule

In this section, we identify which of the basic properties of rules the equal area rule satisfies. These properties are the following.

The $\frac{1}{|N|}$-truncated-claims lower bound on awards[8] says that each claimant should receive at least $\frac{1}{|N|}$th of his claim truncated at the endowment.

Order preservation says that, given two claimants, the award to the larger claimant should be at least as large as the award to the smaller claimant, and that their losses should also be ordered in that way. This property obviously implies the common requirement that two claimants whose claims are equal be assigned equal amounts, **equal treatment of equals**. **Homogeneity** says that multiplying the data of a problem by any $\lambda > 0$ results in a new problem that is solved by rescaling by λ the awards vector chosen for the initial problem.

Endowment monotonicity says that if the endowment increases, each agent should receive at least as much as he did initially. **Order preservation under endowment variations** says that if the endowment increases, given two claimants, the award to the larger claimant should increase by at least as much as the award to the smaller claimant.

Claim monotonicity says that if an agent's claim increases, he should receive at least as much as he did initially. **Bounded award increase under claim increase** says that if an agent's claim increases, his award should not increase by more than his claim did. **Linked claim-endowment monotonicity** says that if an agent's claim and the endowment increase by equal amounts, that claimant's award should not increase by more than that amount.

Claims truncation invariance says that truncating a claim at the en-

[8]The bound is introduced by Moreno-Ternero and Villar (2004) under the name of "securement". *Order preservation* is introduced by Aumann and Maschler (1985), and *order preservation under endowment variations* by Dagan, Serrano, and Volij (1997) under the name of "supermodularity". *Linked claim-endowment monotonicity* appears in connection with a discussion of the duality operator in Thomson and Yeh (2008), and *bounded gain under claim increase* is introduced by Kasajima and Thomson (2012) together with a variety of other monotonicity properties. *Claims truncation invariance* is introduced by Curiel, Maschler, and Tijs (1987) and *minimal rights first* by the same authors under the name of the "minimal rights property". *Composition down* is introduced by Moulin (1987), *composition up* by Young (1988), and duality notions, including *self-duality*, by Aumann and Maschler (1985).

dowment should not affect the awards vector that is selected. **Minimal rights first** says a problem can be equivalently solved in either one of the following two ways: (i) directly; (ii) in two steps, by first assigning to each claimant the difference between the endowment and the sum of the claims of the other claimants, or 0 if this difference is negative, and then the amount he would be assigned in the problem in which claims are reduced by these first-round awards and the endowment by their sum.

Composition down says that if the endowment decreases from some initial value, the awards vector for the new problem can be computed in either one of the following two ways: (i) directly; (ii) by using as claims vector the awards vector calculated for the initial endowment. **Composition up** (Young, 1988) is a counterpart of this invariance property that pertains to possible increases in the endowment.

Self-duality says that the awards vector selected by a rule for some problem is equal to the vector of losses implied by its choice in the "dual" problem, that is, the problem with the same claims vector but an endowment equal to the deficit in the initial problem.

When discussing *claims truncation invariance*, we will refer to the following characterization (Thomson, 2018):

Lemma 2. *For $|N| = 2$, say $N \equiv \{1, 2\}$. A rule S is claims truncation invariant if and only if it can be described in terms of the following networks of paths:*

(a) a path $F \subset \mathbb{R}^N_+$ that, for each $E \in \mathbb{R}_+$, meets the line of equation $x_1 + x_2 = E$ exactly once;

(b1) for each $c_2 \in \mathbb{R}_+$, a path $G(c_2) \subset \mathbb{R}^N_+$ that, for each $E \geq c_2$, meets the line of equation $x_1 + x_2 = E$ exactly once, and is bounded above by the line of equation $x_2 = c_2$;

(b2) for each $c_1 \in \mathbb{R}_+$, a path $G(c_1) \subset \mathbb{R}^N_+$ that, for each $E \geq c_1$, meets the line of equation $x_1 + x_2 = E$ exactly once, and is bounded to the right by the line of equation $x_1 = c_1$; and

(c) for each $c \in \mathbb{R}^N_+$ a path $H(c) \subset \mathbb{R}^N_+$ that, for each $E \in [\max\{c_i\}, c_1 + c_2]$, meets the line of equation $x_1 + x_2 = E$ exactly once, and is bounded above by c,
these paths being used as follows: for each $c \in \mathbb{R}^N_+$ such that $c_1 \geq c_2$, the path for c follows F until the line of equation $x_1 + x_2 = c_2$, then follows $G(c_2)$ until the line of equation $x_1 + x_2 = c_1$, then follows $H(c)$ until c; also for

each $c \in \mathbb{R}^N_+$ such that $c_1 \leq c_2$, the path for c follows F until the line of equation $x_1 + x_2 = c_1$, follows $G(c_1)$ until the line of equation $x_1 + x_2 = c_2$, then follows $H(c)$ until c.

If in addition to *claims truncation invariance*, a rule satisfies *equal treatment of equals*, the path F is the 45° line.

Theorem 1. *The equal area rule satisfies the following properties: The $\frac{1}{|N|}$-truncated-claims lower bound on awards, order preservation, homogeneity, endowment monotonicity, order preservation under endowment variations, claim monotonicity, bounded gain under claim increase, linked claim-resource monotonicity, and claims truncation invariance.*

It violates minimal rights first, composition down, composition up, and self-duality.

Proof. The proofs of most of these statements can be obtained from Lemma 1 by straightforward calculations that we omit.

- The $\frac{1}{|N|}$*-truncated-claims lower bound on awards.* For two claimants, meeting this bound requires each path of awards to contain the segment from the origin to the point whose coordinates are equal to half of the smaller claim. This is what is described under Case 1 of Lemma 1.
- *Order preservation.* Assuming $c_1 \leq c_2$ (and symmetrically if $c_2 \leq c_1$), the path of awards for each $c \in \mathbb{R}^N_+$ should lie on or above the 45° line and on or below the line of slope 1 passing through c. This is easily verified for the equal area rule.
- *Homogeneity.* Again, this property follows directly from the definition of the equal area rule.
- *Endowment monotonicity.* This means that paths of awards should be monotone curves. This is the case for the equal area rule. In fact, the rule satisfies the strict version of this property, which says that as the endowment increases, any claimant whose claim is positive should be assigned more.
- *Order preservation under endowment variation.* Let $c \in \mathbb{R}^N_+$. For a rule whose paths of awards are differentiable curves, this means that if $c_1 < c_2$, the slope of $p^A(c)$ is at least 1. Here, differentiability holds at every point except when the endowment is equal to c_2, and this slope requirement is easily verified.
- *Claim monotonicity.* The equal area rule satisfies this property but not its strict version, which says that, if the endowment is positive, a claimant

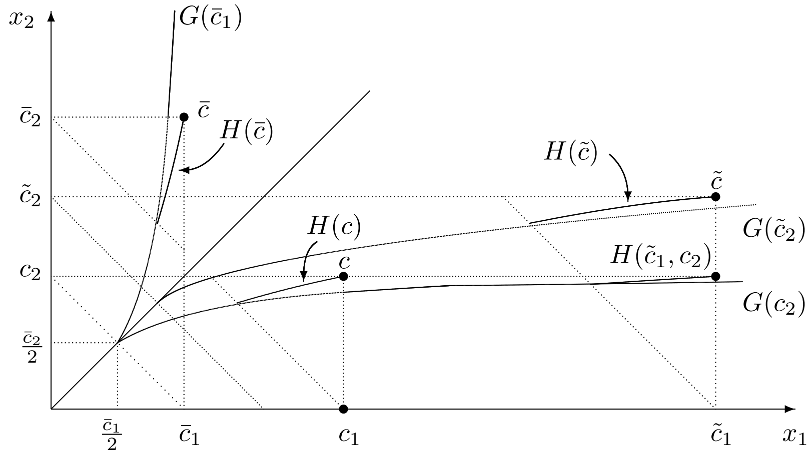

Figure 3: Generating paths of awards of the equal area rule. Keeping agent 2's claim fixed at c_2 and $\tilde{c}_2$, we show the curves $G(c_2)$ and $G(\tilde{c}_2)$. The path for $\tilde{c} \equiv (\tilde{c}_1, \tilde{c}_2)$ consists of some initial segment of the 45° line, a piece of $G(\tilde{c}_2)$ and and a curvi-linear segment $H(\tilde{c})$.

whose claim increases should be assigned more. Indeed, each of its paths of awards starts with a segment of slope 1 that emanates from the origin and whose length is equal to half of the smaller claim (Case 1 of Lemma 1).

• *Bounded gain under claim increase.* Proving that the equal area rule satisfies this property requires more extensive calculations, but they are straightforward as well. We omit them.

• *Linked claim-endowment monotonicity.* Let $x \equiv A(c, E)$. Assuming that c_1 increases by δ, at the point $x+(\delta, 0)$, the sacrifice made by claimant 1 is the same as at x whereas that of claimant 2 is larger. To reestablish equality, claimant 1's award should increase by less than δ.

• *Claims truncation invariance.* This follows directly from the definition of the equal area rule. The curves in terms of which its paths of awards can be described and whose existence is stated in Lemma 2 are $(G(c_1))_{c_1 \in \mathbb{R}_+}$ and $(G(c_2))_{c_2 \in \mathbb{R}_+}$. Given $c \in \mathbb{R}^N_+$ with $c_1 \leq c_2$, the path for c follows the 45° line up to the point of coordinates $(\frac{c_1}{2}, \frac{c_1}{2})$, then it follows $G(c_1)$ until it meets the line of equation $x_1 + x_2 = c_2$. Figure 3 shows a few sample paths of awards.

• *Minimal rights first.* Let $(c, E) \in \mathcal{C}^N$ be given by $c \equiv (4, 8)$ and $E = 8$. Then $A(c, E) = (3, 5)$. The vector of minimal rights in (c, E) is $(8-8, 8-4) = (0, 4)$ and $A(c - (0, 4), 8 - (0 + 4)) = (2, 2)$. Since $A(c, E) \neq (0, 4) + (2, 2)$, the equal area rule violates the property.

• *Composition down.* Let $(c, E) \in \mathcal{C}^N$ be given by $c \equiv (4, 8)$ and $E = 8$. Then $x \equiv A(c, E) = (3, 5)$. Let $E' \equiv 4$. We have $A(c, E') = (2, 2)$. However, the path of the equal area rule for x contains seg$[(0, 0), (\frac{3}{2}, \frac{3}{2})]$ and continues with the portion of the curve $G(3)$ which lies above the 45° line. Thus

$A(x, E') \neq A(c, E')$; the equal area rule violates the property.

• *Composition up.* Let $(c, E) \in \mathcal{C}^N$ be given by $c \equiv (4, 8)$ and $E = 4$. Then $x \equiv A(c, E) = (2, 2)$. Now, let $E' \equiv 8$. We have $A(c, E') = (3, 5)$. However, the path of A for $c - x = A(2, 6)$ contains $\text{seg}[(0, 0), (1, 1)]$ and continues with the strictly monotone curve $G(2)$. Thus, $A(c, E') \neq A(c, E) + A(c - x, E'_E)$ and the equal area rule violates the property. We omit the straightforward derivation.

• *Self-duality.* This property implies that the path of awards for each $c \in \mathbb{R}^N_+$ pass through $\frac{c}{2}$. This is the case only if $c_1 = c_2$.

□

Two rules are dual if for each problem, one rule divides the endowment in the same way as the other divides the shortfall (the difference between the sum of the claims and the endowment) in the problem in which the claims vector is the same but the endowment is equal to the shortfall of the first problem. *Self-duality* is invariance under the duality operator.

It is clear that the equal area rule is not *self-dual*. Its dual is the rule that selects, for each problem $(c, E) \in \mathcal{C}^N$, the awards vector x with the property that among the points that are dominated by c and lie above the budget line, the area of those that are below the line of ordinate c_2 is equal to the area of those that are to the left of the line of abscissa c_1. When generalized to bargaining games in the obvious way (in the above statement, simply replace "lie above the budget line" by "lie above the boundary of the feasible set"), we obtain a solution proposed by Karagözoğlu and Rachmilevitch (2016).

4 Consistency

So far, we have only considered the two-claimant case. For more than two claimants, we begin by noting a difficulty that arises in extending the definition of the equal area rule. To illustrate, let us return to bargaining games. Let $N \equiv \{1, 2, 3\}$ and x be an efficient point of some $S \in \mathcal{B}^N$. In order to evaluate an agent's sacrifice at a proposed compromise, it appears natural to work with volumes. For each $i \in N$, let then $V_i(x, S)$ be the volume of the part of S of all points at which player i's utility is at least as large as x_i. The difficulty comes from the fact that $V_1(S, x)$ and $V_2(S, x)$ typically have a non-empty intersection. In Figure 4, $V_1(S, x)$ is shown to consist of three regions, labeled $W_1(S, x)$, $W_{12}(S, x)$, and $W_{13}(S, x)$. At each point of $W_i(S, x)$,

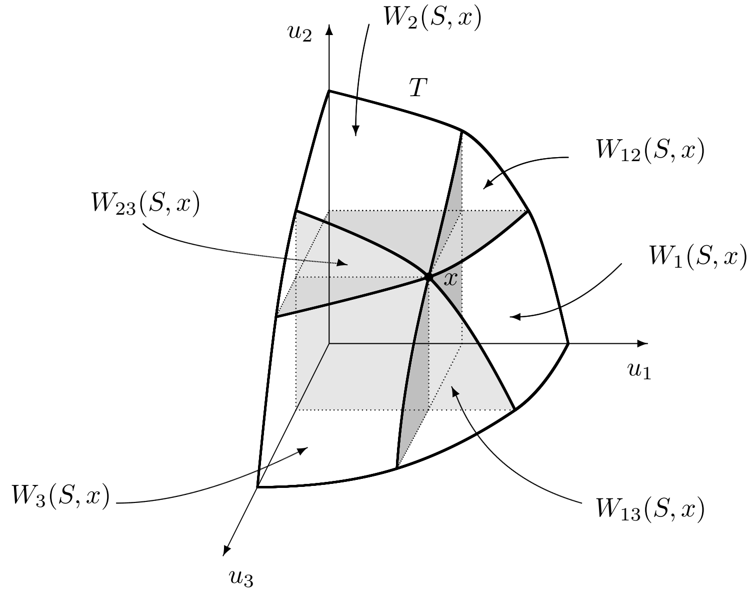

Figure 4: Illustrating the difficulty in generalizing the equal area bargaining solution to more than two players. The region of points at which player i's payoff is at least as large as x_i is denoted $V_i(S,x)$. The regions of points at which two players's payoff are at least as large at a typical point x overlap. For instance, the intersection of $V_1(S,x)$ and $V_2(S,c)$ is $W_{12}(S,x)$.

player i's utility is at least as large as at x and it is the opposite for players j and k. At each point of $W_{ij}(S,x)$, players i and j's utilities are at least as large as at x and it is the opposite for player k. Should we simply look for a point at which all $V_i(S,x)$ are equal? Would ignoring the region $W_{ij}(S,x)$ of overlap when defining the sacrifices made by players i and j at x be unfair to player k? Instead, should this common volume be somehow "shared" between players i and j? A discussion of these various options, and of their pros and cons, is in Thomson (1996).

In our search for an extension of the equal area rule to more than two claimants, we will sidestep the difficulty just discussed and impose a property of coherence of rules across populations of different sizes. For that purpose, we need to generalize our framework of analysis. We imagine that there is an infinite set of "potential" claimants indexed by the natural numbers, $\mathbb{N}$. Let $\mathcal{N}$ be the family of finite subsets of $\mathbb{N}$; these are the populations that may be involved in a claims problem. A rule is now defined over $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$.

Consider the following property of such a rule. Having identified the awards vector it chooses for some problem, we imagine that some claimants leave the scene with their awards and we reevaluate the situation at this point. The amount available for the remaining claimants is equal to the endowment minus the sum of the awards to the claimants who left. Let us apply the rule to this "reduced" problem. **Consistency** says that the rule should choose the same award for each of the remaining claimants as it did initially. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, and—introducing $x \equiv S(c, E)$—we have $x_{N'} = S(c_{N'}, \sum_{N'} x_i) = S(c_{N'}, E - \sum_{N \setminus N'} x_i)$.

It will be convenient to rephrase this requirement by saying that if x belongs to the path of awards of the rule for c, its projection on any coordinate subspace belongs to its path for the projection of c onto the subspace. Thus, its path for c, when projected on that subspace, is a subset of its path for the projection of c. Moreover, if a rule is *endowment continuous*, which is the case for the equal area rule, the projection of its path for c is in fact equal to its path for the projection of c.

Similar questions have been asked about other two-claimant rules. One of them is the rule known as **concede-and-divide**. For each claims vector, this rule is defined by assigning to each claimant i the amount conceded by the other one, claimant j, namely the difference between the endowment and claimant j's claim, or 0 is that difference is negative, and in dividing the remainder equally. It turns out that concede-and-divide has a *consistent* extension, which is none other than the so-called Talmud rule. On the other hand, the rule obtained from the proportional rule by first truncating claims at the endowment has no such extension (Dagan and Volij, 1997).

A general technique to identify the consistent extension of a two-claimant rule when such an extension exists, or to prove that none does if that is the case, is developed in Thomson (2007). It exploits the projection implication of *consistency* just noted. This technique is particularly useful when paths of awards are piece-wise linear, as is often the case, but it has also helped address the question of existence of *consistent* extensions of rules whose paths of awards are not piece-wise linear. For example, it can be used to prove the non-existence, mentioned above, of a *consistent* extension of the version of the proportional rule defined by truncating claims at the endowment first (Thomson, 2008). The proof of the negative result that we offer next follows the same logic.

Theorem 2. *The equal area rule has no consistent extension.*

Proof. Let $N \equiv \{1, 2, 3\}$ and $c \in \mathbb{R}^N_+$ be such that $c_1 < c_2 < c_3$. Because $c_1 < c_2$, $p^A(c_1, c_2)$ includes $\text{seg}[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2})]$ and the part C of the curve $G(c_1)$ in $\mathbb{R}^{\{1,2\}}_+$ that lies between the lines of equation $x_1 + x_2 = c_1$ and $x_1 + x_2 = c_2$.

Similarly, because $c_1 < c_3$, $p^A(c_1, c_3)$ includes $\text{seg}[(0, 0), (\frac{c_1}{2}, \frac{c_1}{2})]$, and the part D of the curve $G(c_1)$ in $\mathbb{R}^{\{1,3\}}$ that lies between the lines of equation $x_1 + x_3 = c_1$ and $x_1 + x_3 = c_3$.

Because A is *strictly endowment monotonic*, $p^A(c_1, c_2)$ and $p^A(c_1, c_3)$ are strictly monotone curves, and one can recover $p^A(c)$ from them as follows. Given $t \in [0, c_1]$, the plane P^t of equation $x_1 = t$ crosses $p^A(c_1, c_2)$ at a single point, x^t, and it crosses $p^A(c_1, c_3)$ at a single point, y^t. There is a unique point $z^t \in \mathbb{R}^N$ whose projections onto $\mathbb{R}^{\{1,2\}}$ and $\mathbb{R}^{\{1,3\}}$ are x^t and y^t respectively. Because the same curve $G(c_1)$ is used to generate $C \subset p^A(c_1, c_2)$ and $D \subset p^A(c_1, c_3)$, it follows that up to an endowment equal to $c_2 = \min\{c_2, c_3\}$, C and D are the same curve (except that one lies in $\mathbb{R}^{\{1,2\}}_+$ and the other in $\mathbb{R}^{\{2,3\}}_+$), so that $x^t_2 = y^t_3$. Thus, the first two coordinates of z^t are equal, and by letting t run from $\frac{c_1}{2}$ to $c_1(1 - \frac{c_1}{2c_2})$, the abscissa of the topmost point of C, we deduce that the path for c of a *consistent* extension of A, if such an extension exists, contains, in addition to $\text{seg}[(0,0,0), (\frac{c_1}{2}, \frac{c_1}{2}, \frac{c_1}{2})]$, a monotone curve in $\mathbb{R}^N$ in the plane of equation $x_1 = x_3$ whose topmost point has second and third coordinates equal to $c_1(1 - \frac{c_1}{2c_2})$. $(*)$ The projection of these two objects onto $\mathbb{R}^{\{2,3\}}$ is $\text{seg}[(0,0), (c_1(1 - \frac{c_1}{2c_2}), c_1(1 - \frac{c_1}{2c_2}))]$.

However, we also know that the path of awards of A for (c_2, c_3) consists of $\text{seg}[(0,0), (\frac{c_2}{2}, \frac{c_2}{2})]$, and that it continues with the part of the curve $G(c_2)$ in $\mathbb{R}^{\{2,3\}}$ that lies between the lines of equation $x_2 + x_3 = c_2$ and $x_2 + x_3 = c_3$. Because $c_1(1 - \frac{c_1}{2c_2}) > \frac{c_2}{2}$, we obtain a contradiction to $(*)$. $\square$

In the face of the negative result stated as Theorem 2, the question arises as to what to do for more than two claimants and preserve the spirit of the equal area rule. The notion of **average consistency** comes to our rescue. A rule satisfies this property if for each problem and each claimant, the award to this claimant is equal to the average of his awards in all of the two-claimant reduced problems associated with it involving him (Dagan and Volij, 1997). Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $x_i = \frac{1}{|N|-1} \sum_{j \in N \setminus \{i\}} S_i(c_i, c_j, x_i + x_j)$. Although the equal area rule has no *consistent* extension, we have the following existence and uniqueness result involving *average consistency*.

Theorem 3. *The equal area rule has a unique average consistent extension.*

Indeed, the only requirement for such an extension of a two-claimant rule to exist, and uniqueness is implied too, is that it be *endowment monotonic* (Dagan and Volij, 1997), and we have seen that the equal area rule enjoys this property.

The operator that associates with each two-claimant rule its *average consistent* extension preserves many of its properties. Included are *endowment monotonicity*, *anonymity* (Dagan and Volij, 1997), *claim monotonicity*, *claims continuity*, and *claims truncation invariance*. Thus, the *average consistent* extension of the two-claimant equal area rule satisfies each of the properties just enumerated.

5 Concluding comments

In certain circumstances, one may decide that a particular claimant is more deserving than some other claimant, independently of the relative values of their claims. For example, one may give preferential treatment to a war veteran and to a single mother. To accommodate this possibility, one can assign weights to claimants and require that rules "respect" or "reflect" these weights. The most natural way to achieve this here is to select, for each problem, a point at which the areas appearing in the original definition, multiplied by the players' respective weights, are equal. For each claims vector, as the relative weights assigned to two claimants go to infinity, the path of awards for that claims vector approaches that of the sequential priority rule in which the claimant who is first is the one who is assigned the greater weight. All of the properties of the equal area rule are preserved under this generalization except, obviously, the $\frac{1}{|N|}$-*lower bound* and all *order preservation* properties. It is indeed the purpose of assigning different weights to claimants to inflect awards in their direction.

An alternative to the equal area bargaining solution that can also be understood as attempting to equate sacrifices among players and has been the object of some discussion is the solution that selects, for each bargaining problem, the point x for which the lengths of the curvi-linear segments in its boundary that connect x to the endpoints of the set of undominated payoff vectors are equal. This "equal length bargaining solution" can be applied to claims problems to generate an "equal length rule". It is an easy matter to check that this rule is none other than the well-studied "concede-and-divide" rule. The same comment applies to the Perles-Maschler bargaining solution (1981). Although these two bargaining solutions generally differ, the rules they induce for claims problem indeed coincide.[9] It is known that concede-

[9]The typical path of awards of this rule for a claims vector $(c_1, c_2) > 0$ contains the same initial segment as the equal area rule, a segment that is symmetric with respect to

and-divide has only one consistent extension, the "Talmud rule", so-called because it rationalizes resolutions proposed in the Talmud for particular numerical examples (Aumann and Maschler, 1985).

Finally, one may argue that instead of measuring agent i's sacrifice at a proposed compromise x by the area of the set of points y at which his award is greater than x_i, the difference between y_i and x_i be taken into consideration. A simple idea would be to measure the sacrifice imposed on claimant $i \in N$ by the integral over $t \in [x_i, c_i]$ of the product $(t - x_i)(E - t)$.

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the half claims vector, and a vertical or horizontal segment connecting these two objects, depending upon whether $c_1 \leq c_2$ or $c_2 \leq c_1$.

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